MATH 307 SPRING 2025
AXIOMS FOR RINGS, EQUALITY, AND THE INTEGERS

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Axioms for Equality

- *Equivalence relation.* You can change formulas using reflexive, symmetric, and transitive properties of $=$. (More formally, you can write down $(\forall x)[x = x]$, $(\forall x)(\forall y)[x = y \Rightarrow y = x]$, and $(\forall x)(\forall y)(\forall z)[(x = y \wedge y = z) \Rightarrow x = z]$.)
- *Substitution.* If s and t are terms, and you know $s = t$, and you have a formula ϕ on some line of the proof, you can write down the result of replacing some or all copies of s by copies of t, as long as no variables in s or t are bound in ϕ.

We also had one more, derived property of equality:

- If you have an equation on some line of the proof, and f is a function, you can write down the result of applying f to both sides of the equation as a new equation.

Ring Axioms

- $+$ is associative
- $+$ is commutative
- $+$ has an identity element (called 0)
- Every element has an additive inverse
- $\times$ is associative
- $\times$ is commutative
- $\times$ has an identity element (called 1)
- Distributive law

Ordered Rings

An *ordered ring* is a ring together with a binary relation (2-place predicate symbol) $<$ such that:

- For all x, y, exactly one of $x < y$, $x = y$, or $y < x$ holds.
- For all x, y, z, $x < y \wedge y < z \Rightarrow x < z$.
- For all x, y, z, $x < y \Rightarrow x + z < y + z$.
- For all x, y, $(0 < x \wedge 0 < y) \Rightarrow 0 < xy$.

The Integers

The integers are an ordered ring $(\mathbb{Z}, +, \times, <)$ with the following extra property. Let $\mathbb{N} = \{x \in \mathbb{Z} \mid 0 < x\}$. The extra property is that $\mathbb{N}$ is well-ordered: any nonempty subset of $\mathbb{N}$ has a smallest element.

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