MATH 307 SPRING 2025
WEEK 9 HOMEWORK
DUE JUNE 2

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Note on your solutions who you worked with, anyone else you got help from, and any materials other than the textbooks that you used. *Electronic resources (Google, ChatGPT, Chegg, etc.) except the textbooks are **not** allowed.*

Required problems (turn these in):

(1) Prove (directly from the ϵ-δ definition) that $f(x) = x^2 - 2x + 1$ is continuous at $x = 3$. Then prove that $f(x)$ is continuous on all of $\mathbb{R}$.

(2) Prove (directly from the ϵ-δ definition) that $f(x) = \sqrt{x}$ is continuous at $x = 4$.

(3) Prove that

$$f(x) = \begin{cases} \frac{1}{(x-2)^2} & x \neq 2 \\ 10 & x = 2 \end{cases}$$

is not continuous at $x = 2$.

(4) Give a precise definition of $\lim_{x\to\infty} f(x) = L$, both in a mostly words form ("For every $\epsilon > 0$...") and purely as a formula ("$(\forall \epsilon)$ ...").

(5) Prove that $\lim_{x\to\infty} \frac{2x+3}{4x+5} = 1/2$.

(6) Write a formula for the negation of the statement that $\lim_{x\to\infty} f(x) = L$. That is, express the statement "it is false that the limit as x goes to infinity of $f(x)$ is L". Don't just stick a $\neg$ in front of your definition of $\lim_{x\to\infty} f(x) = L$.

(7) Write a formula for the statement "$\lim_{x\to\infty} f(x)$ does not exist."

(8) I am making a snowball to throw at an unspecified colleague.[1] For me to be able to hit the colleague, the snowball must approximately 81 grams. The weight W (in grams) of a snowball is given by the following function of its radius r (in cm):

$$W(r) = 3r^3.$$

(This is from estimating snow at a density of $750kg/m^3 = .75g/cm^3$ which a random Internet site[2] seems to think is reasonable, and estimating $\pi = 3$.) So, obviously, I want to make the snowball have radius exactly $3cm$. It is, however, hard to control the snowball's radius exactly.

(a) Find an error tolerance for r that guarantees $W(r)$ will be within $5g$ of the target weight (81g).

(b) Find an error tolerance for r that guarantees $W(r)$ will be within $1g$ of the target weight (81g).

(c) Find an error tolerance for r that guarantees $W(r)$ will be within ϵ of the target weight (81g).

(d) What guarantees that you can always solve the previous part?

[1] I neither endorse nor condone the throwing of snowballs at colleagues.

[2] `https://www.canada-greenhouse-kits.ca/pages/snow-weight`

(9) Copy the following proof onto your own paper and fill in the blanks to prove: If $f(x)$ and $g(x)$ are continuous at a, then so is $h(x) = f(x) + g(x)$.

Thinking. If I know $|f(x) - f(a)| < \epsilon$ and $|g(x) - g(a)| < \epsilon$, then $|h(x) - h(a)| < \boxed{}$.

The proof. Fix $\epsilon > 0$. Then, since $f(x)$ $\boxed{}$, there is a $\delta_1 > 0$ so that $(|x - a| < \delta_1) \Rightarrow (\boxed{} < \epsilon/2)$. Similarly, $\boxed{}$, there is a $\delta_2 > 0$ so that $(|x - a| < \delta_2) \Rightarrow (\boxed{} < \epsilon/2)$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $|x - a| < \delta$, then

$$
\begin{aligned}
|h(x) - h(a)| &= \boxed{} \\
&\leq |f(x) - f(a)| + |g(x) - g(a)| \\
&< \boxed{} = \epsilon.
\end{aligned}
$$

Thus, for any $\boxed{}$ there is a $\boxed{}$ so that $(|x - a| < \delta) \Rightarrow (|h(x) - h(a)| < \epsilon)$. Hence, $h(x)$ is continuous at a, as desired.

(10) Let $f(x) = \frac{x^2 - 9}{x - 3}$. Prove that $\lim_{x \to 3} f(x) = 6$.

(11) Copy the following proof onto your own paper and fill in the blanks to prove the *Squeeze Theorem*: If $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R} \setminus \{a\}$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

Proof. Fix $\epsilon > 0$. Since $\lim_{x \to a} f(x) = L$, there is a $\boxed{}$ so that $(0 < |x - a| < \delta_1) \Rightarrow (|f(x) - L| < \epsilon)$. Similarly, since $\lim_{x \to a} h(x) = L$, there is a $\boxed{}$ so that $\boxed{}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Since $\boxed{}$ (for any $x \neq a$), $f(x) - L < g(x) - L < h(x) - L$. So, $|g(x) - L| < \max\{|f(x) - L|, |h(x) - L|\}$. So, if $0 < |x - a| < \boxed{}$, then

$$
|g(x) - L| < \max\{|f(x) - L|, |h(x) - L|\} < \max\{\boxed{}, \boxed{}\} = \epsilon.
$$

Thus, since $\boxed{}$ and $\boxed{}$ were arbitrary, $\lim_{x \to a} g(x) = L$.

Extra Practice. Do not turn these in, but here are some more optional problems.

(EP-0) Choose some more polynomials and prove they are continuous at specific points, and at any point.

(EP-0) Prove that $f(x) = \frac{1}{x^2+1}$ is continuous on all of $\mathbb{R}$.

(EP-0) Prove that if $f(x)$ is a continuous function and $f(a) \neq 0$ then $g(x) = 1/f(x)$ is continuous at a.

(EP-0) Formulate a version of the squeeze theorem for sequences instead of functions. Prove it.

(EP-0) Write a formula for the statement "$\lim_{x\to a} f(x) = \infty$."

(EP-0) Prove that $\lim_{x\to 2} \frac{1}{(2-x)^2} = \infty$.

(EP-0) Write a formula for the statement "It is false that $\lim_{x\to a} f(x) = \infty$."

(EP-0) Prove that it is not true that $\lim_{x\to 2} x^2 = \infty$.

(EP-0) Prove that it is not true that $\lim_{x\to 2} \frac{1}{2-x} = \infty$.

(EP-0) Write some more story problems like Problem 8.

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