MATH 307 SPRING 2025
WEEK 8 HOMEWORK
DUE MAY 27

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Updated: replaced and errant = by a $\sim$ in problem 2(c,d).

Note on your solutions who you worked with, anyone else you got help from, and any materials other than the textbooks that you used. *Electronic resources (Google, ChatGPT, Chegg, etc.) except the textbooks are **not** allowed.*

Required problems (turn these in):

(1) Turn the following statements into formulas:
  (a) Every positive integer is the sum of four perfect squares.
  (b) Every positive integer not congruent to 7 modulo 8 is the sum of three perfect squares.
  (c) Every positive prime number congruent to 1 modulo 4 is the sum of two perfect squares.
  (d) There is an integer which is not the sum of three squares.
  (e) Every positive real number has a fourth root.
  (f) No negative integer is the sum of three squares.

(2) Let $Q = \{(a, b) \in \mathbb{Z}^2 \mid b \neq 0\}$. Define an relation $\sim$ on Q by $(a, b) \sim (c, d)$ iff $ad = bc$. You will prove that $\sim$ is an equivalence relation. Let $\mathbb{Q}$ denote the set of equivalence classes for $\sim$. The set $\mathbb{Q}$ is called the set of *rational numbers*. (An element $[(a, b)]$ of $\mathbb{Q}$ is usually written as a/b.)
  (a) Prove that $\sim$ is an equivalence relation. (I'm asking for a paragraph proof; it will be pretty short.)
  (b) List 5 elements of the equivalence class of $(3, 2)$.
  (c) Define a multiplication on Q by $(a, b) \times (c, d) = (ac, bd)$. Prove that if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then $(a, b) \times (c, d) \sim (a', b') \times (c', d')$. (So, multiplication gives a well-defined operation on $\mathbb{Q}$.)
  (d) Define an addition on Q by $(a, b) + (c, d) = (ad + bc, bd)$. Prove that if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then $(a, b) + (c, d) \sim (a', b') + (c', d')$.
  (e) What goes wrong if I had allowed $b = 0$?

(3) A relation R on a set S is a *partial order* if R is reflexive, transitive, and satisfies $aRb \wedge bRa \Rightarrow a = b$. Which of the following are partial orders on $\mathbb{N}$? For the ones that are not partial orders, say which property fails and give an example showing that it fails.
  (a) aRb iff $a < b$.
  (b) aRb iff $a \leq b$.
  (c) aRb iff $a \equiv_{12} b$.
  (d) aRb iff $a|b$.

(4) (Sundstrom, Problem 7 page 398). Define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y$ is a rational number.

(a) Prove that $\sim$ is an equivalence relation.
(b) List four different real numbers in the same equivalence class as $\sqrt{2}$.
(c) If a is a rational number, what is the equivalence class $[a]$ containing a?

(For this problem, just think of rational numbers the way you usually do, whatever that is, not the way they're discussed in Problem 2.)

(5) I asserted the following theorem in class:

Theorem C. *The following four statements hold:*

(C-v) If $f(x)$ and $g(x)$ are continuous functions, then $f(x)+g(x)$ is a continuous function.
(C-v) If $f(x)$ and $g(x)$ are continuous functions, then $f(x)g(x)$ is a continuous function.
(C-v) For any real number c, the function $f(x)=c$ is a continuous function.
(C-v) The function $f(x)=x$ is a continuous function.

Using Theorem C:
(a) Prove, by induction, that for any natural number n, $f(x)=x^n$ is a continuous function.
(b) Prove that for any natural number n and any real number c, $f(x)=cx^n$ is a continuous function.
(c) Prove, by induction, that any polynomial is a continuous function.

(6) Let $a_n = 2+\frac{1}{\sqrt{n}}$. Prove that $\lim_{n\to\infty} a_n = 2$.
(7) Let $a_n = 2+\frac{1}{\sqrt{n}}$. Prove that $\lim_{n\to\infty} a_n$ is not 4.
(8) Let $a_n = \frac{n^2+1}{(n+1)^2}$. Prove that $\lim_{n\to\infty} a_n = 1$.
(9) Let $a_n = \frac{(-1)^n n}{n+1}$. Prove that $\lim_{n\to\infty} a_n$ does not exist.

Extra Practice. Do not turn these in, but here are some more optional problems.

(EP-0) Prove that the set $\mathbb{Q}$ from Problem (2) is a ring and, in fact, a field.
(EP-0) Suppose S is a set with n elements. How many different equivalence relations can you put on S?
(EP-0) A *rational function* is a function of the form $p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials (and $q(x)$ is not the constant polynomial 0). Formulate a version of Theorem C that lets you prove that rational functions are continuous at all points a where $q(a)\neq 0$, and prove this statement.
(EP-0) Take some more sequences whose limits you know, and prove those limits are what you expect. Prove they aren't what you don't expect.
(EP-0) Take some sequences that you know do not converge, and prove they have no limits.

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