MATH 307 SPRING 2025
WEEK 7 HOMEWORK
DUE MAY 19

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Note on your solutions who you worked with, anyone else you got help from, and any materials other than the textbooks that you used. *Electronic resources (Google, ChatGPT, Chegg, etc.) except the textbooks are* ***not*** *allowed.*

Required problems (turn these in):

(1) Use induction or strong induction to prove Cassini's identity for Fibonacci numbers,

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}.$$

(This is the same as a problem from last week, so you've already done the main work. The new part is phrasing this in terms of induction instead of using WOA.)

(2) Let

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \psi = \frac{1-\sqrt{5}}{2} = 1 - \phi = -\frac{1}{\phi}.$$

Use induction to prove that $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$. (Hint: first show that $\phi^n = \phi^{n-1} + \phi^{n-2}$, and similarly for ψ. Try first the case $n = 2$.)

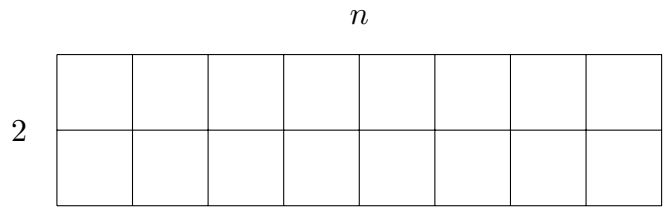
(3) The formula for $\binom{n}{k}$ looks like a fraction, but in fact $\binom{n}{k}$ is an integer. Prove this by induction on n, using the Pascal's Triangle relation.

(4) Prove that if p is prime and $1 \leq k < p$, then $p|\binom{p}{k}$. (Hint: prove that p divides the numerator and use property P to show that p does not divide the denominator. Since $\binom{p}{k}$ is an integer, you can factor the numerator as the denominator times $\binom{p}{k}$. Apply property P one more time to get the result.) You may not have phrased it that way, but part of your argument uses induction—where?

(5) Prove that (unlike over $\mathbb{R}$) if p is prime, then in $\mathbb{Z}/p$, $(x+y)^p = x^p + y^p$.

(6) Let R be a field and $p(x)$ a (nonzero) polynomial of degree n over R. Use induction to prove that $p(x)$ has at most n different roots. (A *root* is an element $a \in R$ so that $p(a) = 0$.) (Hint: Use results from last week's homework.)

(7) Consider an $n \times 2$ grid:



How many ways are there to cover this grid by dominoes, i.e., by 2×1 or 1×2 rectangles? Use induction (or strong induction, or the well-ordering axiom) to prove your answer. (Suggestion: think about some small cases first.)

(8) Find a formula for the number of closed intervals with integer endpoints (including ones of the form $[k, k]$) contained in $[1, n]$. For example, for $[1, 2]$, there are three: $[1, 1]$, $[2, 2]$, and $[1, 2]$. Use induction (or its cousins) to prove your formula. (Suggestion: again, list all the intervals in some small cases to get a feel for the problem.) (This problem is from D'Angelo and West, problem 3.33 on page 73.)

Extra Practice. Do not turn these in, but here are some more optional problems, leading to a proof that in $\mathbb{Z}/p$ there is an element of multiplicative order p. I will probably not post solutions, but am happy to help you with any of them.

(EP-0) Suppose that a and b are nonzero elements of $\mathbb{Z}/p$, and the multiplicative orders of a and b are m and n, respectively. Let ℓ be the least common multiple of m and b. Prove that the multiplicative order of ab is ℓ.

(EP-0) Use the previous problem to show: given a prime p, there is a number n so that the multiplicative order of every nonzero element of $\mathbb{Z}/p$ divides n, and there is some element of multiplicative order n.

(EP-0) Use Fermat's Little Theorem to show that the number n from Problem (EP-2) is at most $p - 1$.

(EP-0) Use Problem (6) to prove that the number n from Problem (EP-2) is at least $p - 1$. (Hint: consider the polynomial $x^{p-1} - 1$.)

(EP-0) Combining the previous two problems, deduce that there is an element of $\mathbb{Z}/p$ of multiplicative order $p - 1$.

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