MATH 307 SPRING 2025
WEEK 6 HOMEWORK
DUE MAY 12

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Note on your solutions who you worked with, anyone else you got help from, and any materials other than the textbooks that you used. *Electronic resources (Google, ChatGPT, Chegg, etc.) except the textbooks are* ***not*** *allowed.*

Updated May 10: Typo in Problem 8 corrected.

Required problems (turn these in):

Use the Well Ordering Axiom to prove:

(1) The sum of the first n perfect squares is $\frac{n(n+1)(2n+1)}{6}$, i.e.,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

More formally,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(2) If F_n denotes the n^{th} Fibonacci number, then

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}.$$

More formally,

$$\sum_{k=0}^{n-1} F_{2k+1} = F_{2n}.$$

(3) Cassini's identity for Fibonacci numbers,

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}.$$

(4) The greatest common divisor of F_n and F_{n+1} is 1.

(5) For any natural numbers $n \geq m$, the binomial coefficients satisfy:

$$\binom{n+1}{m+1} = \binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{n}{m}.$$

More formally,

$$\binom{n+1}{m+1} = \sum_{k=m}^{n} \binom{k}{m}$$

(6) For any natural number n, the binomial coefficients satisfy:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

More formally,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(Note: you might know other proofs of some of these facts, but the goal here is to practice using the WOA.)

(7) Consider the polynomials $f(x) = 1 + 2x + 3x^2$ and $g(x) = 4 + 2x$.
  (a) If we view the coefficients as elements of $\mathbb{R}$, what are $f(x) + g(x)$ and $f(x)g(x)$?
  (b) If instead we view the coefficients as elements of $\mathbb{Z}/5$, what are $f(x) + g(x)$ and $f(x)g(x)$?

(8) *Theorem.* (Division Theorem for Polynomials) Let R be a field and $f(x)$ and $g(x)$ be polynomials whose coefficients are elements of R, with $g(x) \neq 0$. (That is, $g(x)$ is not the zero polynomial; there might be some solutions to the equation $g(x) = 0$.) Then there are polynomials $q(x)$ and $r(x)$ so that $f(x) = q(x)g(x) + r(x)$ and either $r(x) = 0$ or the degree of $r(x)$ is smaller than the degree of $g(x)$.
  (a) We can view $f(x) = x^4 + 1$ and $g(x) = x + 1$ as polynomials with coefficients in $\mathbb{R}$, polynomials with coefficients in $\mathbb{Z}/2$, and polynomials with coefficients in $\mathbb{Z}/3$. Divide $f(x)$ by $g(x)$ in each of these three cases. (The answers will be different.)
  (b) Imitate the proof of the division theorem for natural numbers that we gave in class to prove the division theorem for polynomials. (Hint: let $\deg(f(x))$ denote the degree of $f(x)$. Consider $S = \{\deg(f(x) - q(x)g(x)) \mid q(x) \text{ is a polynomial}\}$, where we define the degree of the zero-polynomial to be -1, say. (Really, we should set $\deg(0) = -\infty$, but that's confusing.))

(9) Using polynomial division, prove that if $f(x)$ is a polynomial, $a \in \mathbb{R}$, and $f(a) = 0$, then you can factor out $(x - a)$ from $f(x)$. Also, give an example of this for polynomials with coefficients in $\mathbb{R}$, and for polynomials with coefficients in $\mathbb{Z}/2$.

(10) Here is a proof that $(\forall a)(\forall n)[a \in \mathbb{N} \wedge n \in \mathbb{Z} \wedge n \geq 0 \Rightarrow a^n = 1]$, i.e., that a^n is always equal to 1. First, notice that $a^0 = 1$, so this is true for $n = 0$. Now, let $S = \{n \in \mathbb{N} \mid a^n \neq 1\}$. Assume that $S \neq \emptyset$ (for indirect inference). Then S has a least element; call it ℓ. Then

$$a^\ell = \frac{a^{\ell-1}a^{\ell-1}}{a^{\ell-2}}.$$

But since ℓ was the least element of S, the right side is just $\frac{(1)(1)}{1}$. So, $a^\ell = 1$, a contradiction. Thus, $S = \emptyset$, so $a^n = 1$ for all n. Since a was arbitrary, this proves the result.

What is wrong with this argument? (Thanks to Daniel W. Farlow on StackExchange for making me aware of this cute example.)

No extra practice problems this week, but you can always ask me for more practice (or look up some more identities for Fibonacci numbers or binomial coefficients and try to prove them).

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