MATH 307 SPRING 2025
WEEK 3 HOMEWORK
DUE APRIL 21

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Note on your solutions who you worked with, anyone else you got help from, and any materials other than the textbooks that you used. *Electronic resources (Google, ChatGPT, Chegg, etc.) except the textbooks are **not** allowed.*

Required problems (turn these in):

In this homework we will use many times the following definitions applicable to integers a, b and a positive integer n: (1) $a|b \Leftrightarrow (\exists k)(k \in \mathbb{Z} \wedge b = ka)$, and (2) $(a \equiv_n b) \Leftrightarrow (n|(a-b))$ ($a \equiv_n b$ in Definition (2) is the same as $a \equiv b \pmod{n}$, but it looks better typographically).

(1) Show that $[U \wedge P] \Rightarrow [Q \wedge R]$, $P \Leftrightarrow [S \vee T]$, $R \wedge T \ \vdash \ U \Rightarrow Q$.

(2) Show that $\neg P \Rightarrow Q$, $P \Rightarrow \neg Q$, $P \Leftrightarrow R \ \vdash \ \neg Q \Leftrightarrow R$. (Hint: Recall that $X \Leftrightarrow Y$ is an abbreviation for something.)

(3) Translate each of the following into a normal English sentence. Also, identify the statement as True or False. For example, the proposition

$$(\forall x)[(x \in \mathbb{Z} \wedge x|24) \Rightarrow x|48]$$

says that "Every divisor of 24 is a divisor of 48". It is True.

- (c) $(\forall x)[(x \in \mathbb{N} \wedge x|13) \Rightarrow x \in \{1, 13\}]$
- (c) $(\exists a)[a \in \mathbb{N} \wedge (a > 51 \wedge a < 52)]$
- (c) $(\forall x)[(x \in \mathbb{N} \wedge x^4 < 0) \Rightarrow x = 3]$
- (c) $(\forall x)[x \in \mathbb{N} \Rightarrow (\exists y)[y \in \mathbb{N} \wedge y|x]]$
- (c) $(\forall x)(\forall y)[(x \in \mathbb{N} \wedge y \in \mathbb{N}) \Rightarrow ((\exists z)(z > xy))]$

(4) The phrase "the integer x is a perfect square" means $x \in \mathbb{Z} \wedge (\exists y \in \mathbb{Z})[x = y^2]$. Keeping this in mind, write out mathematical statements—using only quantifiers and other math symbols, no English words—which say the same thing as the following sentences. Do not worry about whether the statements are true or false.

- (d) For every integer a, if a is even then a^2 is a multiple of 5.
- (d) 7 is smaller than every integer.
- (d) Every integer which is a perfect square is also a perfect cube.
- (d) No even number is divisible by 13.
- (d) There exists an integer which is larger than every other integer.
- (d) There is an element of the set A having the property that every element of the set B divides it.
- (d) There exists an integer p with the following properties: p is even and for every pair of integers a and b, if p divides ab then it must divide either a or b.

(5) In each part, identify the given set by listing all of its elements. For example:

Given the set $\{x \in \mathbb{Z} \mid x \equiv_4 3 \wedge 5 \leq x \wedge x \leq 20\}$ you would answer $\{7, 11, 15, 19\}$, as these two sets are equal.

- (e) $\{x \in \mathbb{Z} \mid |x| < 10 \wedge 3|x\}$

(e) $\{x^3 + 1 \mid x \in \mathbb{N}\} \cap \{y \mid y \in \mathbb{N} \wedge 1 \leq y \leq 30\}$
(e) $\{x \in \mathbb{N} \mid (\exists a)(\exists b)[a \in \mathbb{N} \wedge b \in \mathbb{N} \wedge x = a^2 + b^2]\} \cap \{x \mid x \in \mathbb{N} \wedge x \leq 20\}$
(e) $\{a \mid a \in \mathbb{Z} \wedge a \equiv_4 3\} \cap \{b \mid b \in \mathbb{Z} \wedge b \equiv_2 0\}$
(e) $\{x \in \mathbb{N} \mid x \equiv_5 1 \wedge x \leq 40\} \cap \{x \in \mathbb{N} \mid x \equiv_6 4\}$

(6) In each part below, use mathematical notation to write the negation of the given statement in such a way that no quantifier is immediately preceded by a negation sign, every universal quantifier is applied to a conditional (statement with $\Rightarrow$), and every existential quantifier is applied to a conjunction (statement with $\wedge$). In each part, decide which statement is true: the given statement or its negation.
(f) $(\forall y)[y \in \mathbb{Z} \Rightarrow y > 0]$
(f) $(\forall x)[(x \in \mathbb{Z} \wedge 3|x) \Rightarrow 6|x]$
(f) $(\forall x)((x \in \mathbb{Z} \wedge x > 0) \Rightarrow (\exists y)(y \in \mathbb{Z} \wedge y < x \wedge y > 0))$
(f) $(\forall x)[x \in \mathbb{Z} \Rightarrow (\exists y)[y \in \mathbb{Z} \wedge x = y^2]]$

(7) Copy the following outline onto your own paper and fill in the blanks in the to prove that $(\forall a)((a \in \mathbb{Z} \wedge a \equiv_2 1) \Rightarrow a^2 \equiv_4 1)$.

(1)	Assume $a \in \mathbb{Z}$ and $a \equiv_2 1$	
(2)		Definition of $\equiv_2$
(3)		Definition of $\mid$
(4)	$a - 1 = 2k$ for some $k \in \mathbb{Z}$	
(5)	$(a-1)^2 = 4k^2$	algebra
(6)	$a^2 =$	algebra
(7)	$2a = 4k + 2$	algebra
(8)	$a^2 - 1 =$	algebra
(8.5)		
(9)		Definition of $\mid$
(10)	$a^2 \equiv_4 1$	
(11)		DT for Hyp (1)
(12)		

(8) Write the previous proof in paragraph form.

(9) Copy the following outline onto your own paper and fill in the blanks to prove that $(\forall a)(\forall b)(\forall k)[[a, b, k \in \mathbb{Z} \wedge (k \geq 1 \wedge a|b)] \Rightarrow a^k|b^k]$. (In this proof, we have left out most of the justifications. Make sure you could fill them in if asked.)

(1)	Assume $a, b, k \in \mathbb{Z}$ and $k \geq 1$ and $a \mid b$.	Hyp for DT
(2)	$\boxed{\quad\quad}(y \in \mathbb{Z} \wedge b = y \cdot a)$	
(3)	$b = y \cdot a$ for some $\boxed{\quad\quad}$	
(4)	$b^k = \boxed{\quad\quad}$	
(5)	$b^k = \boxed{\quad\quad}$	
(6)	$(\exists u)(u \in \mathbb{Z} \wedge \boxed{\quad\quad})$	
(7)	$a^k \mid b^k$	
(8)	$\boxed{\quad\quad}$	DT For Hyp 1
(9)	$(\forall k)[[a, b, k \in \mathbb{Z} \wedge (k \geq 1 \wedge a \mid b)] \Rightarrow a^k \mid b^k]$	
(10)	$\boxed{\quad\quad}$	
(11)	$(\forall a)(\forall b)(\forall k)[[a, b, k \in \mathbb{Z} \wedge (k \geq 1 \wedge a \mid b)] \Rightarrow a^k \mid b^k]$.	

Answer the following questions:

(i) The above proof used one IE step, one EG step, and three UG steps. Label them in column 2 of your proof (the same column with DT in it).

(i) The definition of $a \mid b$ used k for the variable in the existential statement. In step 2, why did we change and use y instead?

(i) In step 5 we introduced the variable u in the existential statement. Could we have used k here instead? What about y?

(10) Write the previous proof in paragraph form.

(11) Prove: $(\forall a)(\forall b)(\forall c)[(a, b, c \in \mathbb{Z} \wedge a \mid b \wedge b \mid c) \Rightarrow a \mid c]$. (Your proof should be in three columns, like the fill-in-the-blank ones.)

(12) Prove: $(\forall a)((a \in \mathbb{Z} \wedge \neg(2 \mid a)) \Rightarrow \neg(12 \mid a))$. (Again, a three column proof.)

Extra Practice. Do not turn these in, but if you had trouble with some of the required arguments, here are some more to practice, for which I will post solutions.

(EP-0) Show that $R \vee S$, $\neg P$, $Q \vee \neg R$, $P \Leftrightarrow Q \;\vdash\; S$.

(EP-0) In each part below, use mathematical notation to write the negation of the given statement in such a way that no quantifier is immediately preceded by a negation sign, every universal quantifier is applied to a conditional (statement with $\Rightarrow$), and every existential quantifier is applied to a conjunction (statement with $\wedge$). In each part, decide which statement is true: the given statement or its negation.

(a) $(\exists a)[a \in \mathbb{N} \wedge (\forall b)[[b \in \mathbb{N} \wedge a \neq b] \Rightarrow b > a]]$
(b) $(\exists m)(\exists n)[m, n \in \mathbb{N} \wedge m > n]$
(c) $(\forall m)(\forall n)[m, n \in \mathbb{N} \Rightarrow m > n]$
(d) $(\exists m)[m \in \mathbb{N} \wedge (\forall n)[n \in \mathbb{N} \Rightarrow m > n]]$
(e) $(\forall m)[m \in \mathbb{N} \Rightarrow (\exists n)[n \in \mathbb{N} \wedge m > n]]$
(f) $(\forall a)(\forall b)[(a, b \in \mathbb{R} \wedge a < b) \Rightarrow (\exists c)[c \in \mathbb{R} \wedge [a < c \wedge c < b]]]$

(EP-0) Translate each of the following into a normal English sentence. Also, identify the statement as True or False.

(a) $(\forall x)[(x \in \mathbb{N} \wedge x^2 = 2) \Rightarrow x = 5]$
(b) $(\forall n)[(n \in \mathbb{N} \wedge (4|n \wedge 6|n)) \Rightarrow 24|n]$

(EP-0) Write out mathematical statements—using only quantifiers and other math symbols, no English words—which say the same thing as the following sentences. Do not worry about whether the statements are true or false.

(a) Every nonzero integer is either a multiple of six or a multiple of seven.
(b) Every element of the set A is either even or a multiple of 13.
(c) There exists an integer which is not divisible by any divisor of 240.
(d) For all integers x, if x is congruent to three mod eight then there exists an integer y such that $x \cdot y$ is congruent to one mod eight.

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