MATH 307 SPRING 2025
WEEK 2 HOMEWORK
DUE APRIL 14

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Solutions are due by Canvas upload at 11:59 p.m. on Monday, April 14. *Please make sure your solutions are easy to read.* Note on your solutions who you worked with, anyone else you got help from, and any materials other than the textbooks that you used. *Electronic resources (Google, ChatGPT, Chegg, etc.) except the textbooks are **not** allowed.*

Required problems (turn these in):

(1) Exner, problem 5, p. 83: Use GSP where appropriate to show that $Q \Leftrightarrow S,\ R \Leftrightarrow \neg S,\ P \Leftrightarrow R\ \vdash\ P \Leftrightarrow \neg Q$.

(2) Show that you can write formulas using just the connective *nor* (which we'll write $\downarrow$) that have the same truth tables as $\neg$, $\wedge$, and $\vee$. (This is really three problems, one for $\neg$, one for $\wedge$, and one for $\vee$.)

(3) The book introduces the *alternative denial* $|$ (Exner, p. 89). Give a disjunctive normal form formula with the same truth table as $|$.

(4) Exner, problem 3, p. 113: Fill in the reasons in the following demonstration of $\neg Q \Rightarrow (R \vee S),\ S \Leftrightarrow (P \vee R),\ \neg P\ \vdash\ \neg R \Rightarrow Q$. As always, copy the whole thing onto your own paper; don't write here.

	Statement	Explanation
(1)	$\neg Q \Rightarrow (R \vee S)$	???
(2)	$S \Leftrightarrow (P \vee R)$	???
(3)	$\neg P$	???
(4)	$\neg R$	???
(5)	$\neg P \wedge \neg R$	???
(6)	$S \Rightarrow (P \vee R)$	???
(7)	$\neg(P \vee R) \Rightarrow \neg S$	???
(8)	$(\neg P \wedge \neg R) \Rightarrow \neg(P \vee R)$	???
(9)	$(\neg P \wedge \neg R) \Rightarrow \neg S$	???
(10)	$\neg S$	???
(11)	$\neg R \wedge \neg S$	???
(12)	$\neg(R \vee S) \Rightarrow Q$	???
(13)	$(\neg R \wedge \neg S) \Rightarrow \neg(R \vee S)$	???
(14)	$(\neg R \wedge \neg S) \Rightarrow Q$	???
(15)	Q	???
(16)	$\neg R \Rightarrow Q$	???

(5) Fill in the following demonstration (copying it onto your own paper), using the Deduction Theorem, to show that $P \Rightarrow Q,\ (\neg P \vee R) \Rightarrow S\ \vdash\ Q \vee S$:

	Statement	Explanation
(1)	$P \Rightarrow Q$	???
(2)	???	Hyp
(3)	$\neg S$	???
(4)	???	MT (2), (3)
(5)	???	Taut
(6)	$P \wedge \neg R$	???
(7)	???	CS (6)
(8)	???	MP (1), (7)
(9)	???	Deduction Theorem for Hyp (3)
(10)	$(\neg S \Rightarrow Q) \Leftrightarrow (Q \vee S)$	???
(11)	$Q \vee S$	???

(6) Fill in the blanks (copying it all onto your own paper) to give a proof of $R \vee [P \wedge Q]$, $\neg Q \ \vdash \ R$.

	Statement	Explanation
(1)	$R \vee [P \wedge Q]$	???
(2)	$\neg R$	???
(3)	???	Taut
(4)	$\neg R \Rightarrow [P \wedge Q]$	???
(5)	$P \wedge Q$	???
(6)	???	CS ???
(7)	???	Hyp
(8)	$Q \wedge \neg Q$	???
(9)	R	???

(7) Fill in the blanks (copying it all onto your own paper) to give a proof of $(P \wedge \neg Q) \Rightarrow (R \Rightarrow Q) \ \vdash \ P \Rightarrow (Q \vee \neg R)$. (Note: This proof uses both DT and Indirect Inference.)

	Statement	Explanation
(1)	$(P \wedge \neg Q) \Rightarrow (R \Rightarrow Q)$	???
(2)	P	Hyp for DT
(3)	$\neg[Q \vee \neg R]$	Hyp for II
(4)	???	Taut
(5)	$\neg Q \wedge R$	???
(6)	$\neg Q$	???
(7)	$P \wedge \neg Q$	???
(8)	$R \Rightarrow Q$	???
(9)	$\neg R$	???
(10)	R	???
(11)	$R \wedge \neg R$	???
(12)	$Q \vee \neg R$	???
(13)	$P \Rightarrow (Q \vee \neg R)$	???

(8) Exner Problem 10 p. 134: Use Indirect Inference to show that $R \Rightarrow \neg Q,\ \neg(S \wedge \neg R) \ \vdash\ Q \Rightarrow \neg S$.

(9) Exner Problem 11 p. 134: Use Indirect Inference to show that $\neg P \Rightarrow Q,\ Q \Rightarrow (R \Rightarrow S),\ \neg S \ \vdash \ R \Rightarrow P$.

(10) The following proof has a mistake. Find what is wrong, and explain.
$(R \vee \neg S) \Rightarrow \neg P,\ Q \Rightarrow R,\ S \Rightarrow T \ \vdash\ (P \Rightarrow \neg R) \wedge (Q \Rightarrow T)$.

	Statement	Explanation
(1)	$(R \vee \neg S) \Rightarrow \neg P$	Hyp
(2)	$Q \Rightarrow R$	Hyp
(3)	P	Hyp for DT
(4)	$\neg(R \vee \neg S)$	MT (1), (3)
(5)	$\neg(R \vee \neg S) \Leftrightarrow (\neg R \wedge S)$	Taut
(6)	$\neg R \wedge S$	MPB (5), (4)
(7)	S	CS (6)
(8)	$\neg R$	CS (6)
(9)	$P \Rightarrow \neg R$	DT for Hyp (3)
(10)	Q	Hyp for DT
(11)	$S \Rightarrow T$	Hyp
(12)	T	MP (11), (7)
(13)	$Q \Rightarrow T$	DT for Hyp (10)
(14)	$(P \Rightarrow \neg R) \wedge (Q \Rightarrow T)$	CI (9), (13).

(11) A set of hypotheses (premises) is *inconsistent* if you can use them to derive a contradiction, and *consistent* if you can't.

Explain why a set of hypotheses is consistent if and only if there's a way to assign truth values to the variables so that the hypotheses are all true. (Hint: since you're justifying an "if and only if," there are two parts: explaining that if the hypotheses are consistent, then there's a way to assign truth values etc.; and explaining the converse, that if there's a way to assign truth values etc. then the hypotheses are consistent. For the second of these, it's easier to argue the contrapositive. Also, you'll want to use the Soundness Theorem and the Completeness Theorem or the Deduction Theorem somewhere in your explanation. This problem is a bit of a stretch for this point in the course; do your best.)

(12) As we saw in class and in the book and above, one can misuse the Deduction Theorem in the middle of a proof, by reusing a statement that depended on the hypotheses assumed only to apply DT. Give an English example of an argument in this style. (The more baroque the better.)

(13) In Imaginary World X, there are two kinds of sentient beings: ducks and beavers. In this world, ducks always tell the truth, and beavers always lie. Other than that, it is impossible to tell them apart—they look and sound identical.

You find yourself visiting IWX, where you meet three locals, A, B, and C.

- A says to you: "B and C are either both ducks or both beavers."
- B says to you: "Both C and I are ducks."
- C says to you: "B is a beaver."

Can you determine whether each of A, B, and C is a duck or beaver? Justify your answer. (Adapted from Rubin, *Mathematical Logic: Applications and Theory*, Problem 3.E.(c).)

(14) I mentioned *Goldbach's Conjecture* in class: every even number bigger than 2 is the sum of two prime numbers. Verify by hand that this holds for all (even) numbers up to 30.

(15) Given two positive integers m and n, their *greatest common divisor* is the largest integer k which divides both m and n evenly. We will write the greatest common divisor as $\gcd(m,n)$. For example, $\gcd(4,6)=2$, and $\gcd(36,120)=12$. Here are two ways to find $\gcd(m,n)$:
- Write m and n as products of powers of primes. For example, $120 = 2^3\cdot 3\cdot 5$ and $36 = 2^2\cdot 3^2$. Now, for each prime, take the smaller of the two powers you see, and multiply these new prime powers together. In our example, that's $2^2\cdot 3^1\cdot 5^0 = 12$.
- Let's suppose $m\leq n$; otherwise, exchange which is called m and which is called n. Let r be the remainder when you divide n by m. If $r=0$ then m divides n so $\gcd(m,n)=m$. Otherwise, $\gcd(m,n)=\gcd(k,m)$. Repeat this process with (k,m) in place of (m,n), and so on, until eventually you're in the first case. For example:
$$\begin{aligned} 120 &= 3\cdot 36+12 && \text{Remainder is 12}\\ 36 &= 3\cdot 12+0 && \text{Remainder is 0.}\end{aligned}$$
So, $\gcd(36,120)=\gcd(12,36)=12$. As another example, to find $\gcd(99,121)$:
$$\begin{aligned} 121 &= 1\cdot 99+22 && \text{Remainder is 22}\\ 99 &= 4\cdot 22+11 && \text{Remainder is 11}\\ 22 &= 2\cdot 11 && \text{Remainder is 0.}\end{aligned}$$
So, $\gcd(99,121)=11$. (This is called the *Euclidean algorithm*.)

Now:

(a) Use both methods to compute $\gcd(100,126)$.

(b) This problem should leave you with some questions. What are you wondering? If you're not wondering anything, what do methods assume / what should we check about them?

Extra Practice. Do not turn these in, but if you had trouble with some of the required arguments, here are some more to practice, for which I will post solutions.

(EP-0) Show that $P\Rightarrow Q,\ R,\ R\Rightarrow [Q\Rightarrow P]\ \vdash\ P\Leftrightarrow Q$.

(EP-0) Show that $P\Rightarrow\neg Q,\ \neg R\Rightarrow Q\ \vdash\ P\Rightarrow R$.

(EP-0) Show that $R\Rightarrow T,\ \neg T\Leftrightarrow S,\ [R\wedge\neg S]\Rightarrow\neg Q\ \vdash\ R\Rightarrow\neg Q$.

(EP-0) Show that $\neg P\Rightarrow Q,\ [R\Rightarrow Q]\Rightarrow S,\ \neg S\vee T,\ R\Rightarrow\neg P\ \vdash\ T\vee V$.

(EP-0) Show that $(R\vee\neg S)\Rightarrow\neg P,\ Q\Rightarrow R,\ S\Rightarrow T\ \vdash\ (P\Rightarrow S)\wedge(Q\Rightarrow(\neg P\wedge R))$.

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