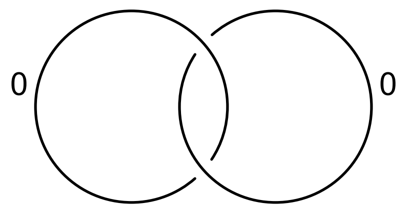
MATH 692 SPRING 2024
HOMEWORK 2
DUE MAY 18, 2024.

INSTRUCTOR: ROBERT LIPSHITZ

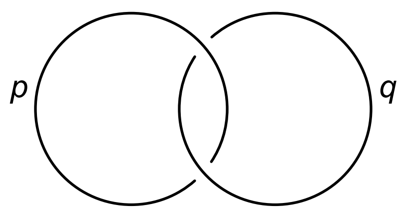
Solve any five of these problems. (Problems marked with stars are also available as mini-paper topics.)

(0) Consider a Kirby diagram consisting entirely of 2-handles, say, with integer framings. What is the homology of the 4-manifold-with-boundary it specifies? Of the 3-manifold it specifies?

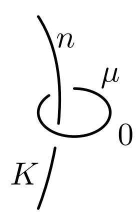
(0) What closed 4-manifold does the following Kirby diagram represent?



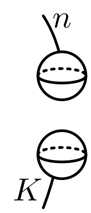
(0) What 3-manifold does the following Kirby diagram represent?



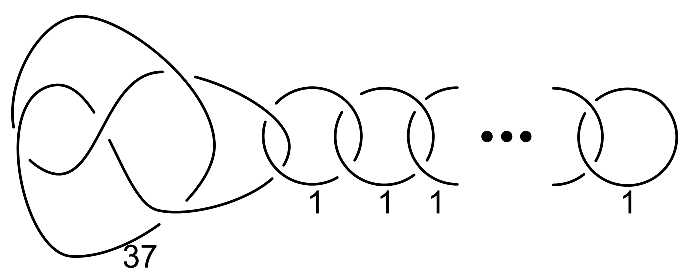
(0) *Suppose a Kirby diagram contains a knot K (with some integral framing) and a meridian μ of K with framing 0:



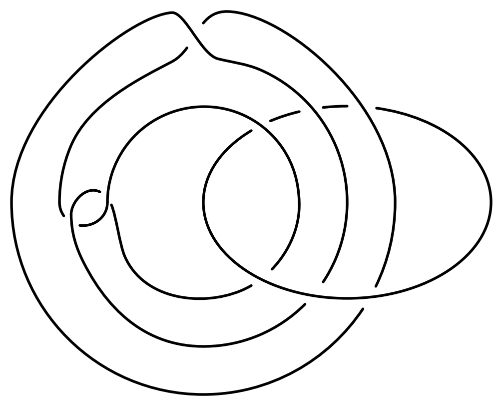
Prove that erasing both K and μ gives a new Kirby diagram representing the same 3-manifold. (Hint: there are several ways to do this. One is directly from the definition of Dehn surgery, thinking about what μ represents after performing surgery on K. Another is to observe that replacing μ by a 1-handle as in the picture below gives the same 3-manifold. A third is to observe that you can use handleslides over μ to unknot K and unlink K from the rest of the Kirby diagram, and then to change the framing of K to 0 or 1 and use a special case of Exercise (3).)



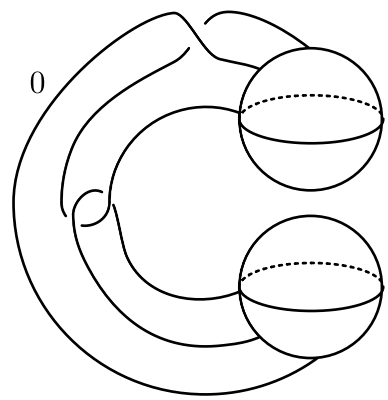
(0) Given Kirby diagrams for 3-manifolds Y_1 and Y_2, explain how to produce a Kirby diagram for $Y_1 \# Y_2$.
(0) Given Kirby diagrams for 4-manifolds-with-boundary W_1 and W_2, explain how to produce a Kirby diagram for their boundary sum $W_1 \natural W_2$.
(0) Given Kirby diagrams for 4-manifolds-with-boundary W_1 and W_2, explain how to produce a Kirby diagram for their connected sum $W_1 \# W_2$.
(0) Find Kirby diagrams with integer surgery coefficients (framings) representing the lens spaces $L(3,2)$ and $L(7,3)$.
(0) Consider the Kirby diagram:



Here, there are 2024 1-framed unknots hanging off, in a row. What 3-manifold does this represent?
(0) Problems 1 or 2 in Chapter 3 of Saveliev's book. We haven't defined the Brieskorn spheres $\Sigma(2,3,5)$ and $\Sigma(2,3,7)$, but he wants you to take the plumbing diagram on the top row of his Figure 19 as the definition of $\Sigma(2,3,7)$, and the first diagram 3.15 as the definition of $\Sigma(2,3,5)$. (The two pictures in the top row of Figure 19 are two ways of writing the same Kirby diagram. It's a bit confusing that he's exchanged the labels -2 and -3 in them.)
(0) Show that there is an involution of S^3 exchanging the components of the following link:



(Hint: draw the link in a more symmetric way.) Use that to construct an involution of the boundary of the following 4-manifold:



(This is an example of an *Akbulut cork*. It turns out that the involution you constructed does not extend smoothly over the 4-manifold itself, and this can be used to construct exotic smooth structures.)

(0) Compute the homology of the 4-manifold described in the previous problem, and of its boundary.

(0) *Consider the Kirby diagram consisting of just an n-framed unknot. Show that it represents the D^2-bundle over S^2 with Euler number n. In particular, for $n = \pm 1$ this Kirby diagram represents $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$ (or the complement of a 4-ball inside them.)

(0) Consider a handle decomposition for a (closed, say) 4-manifold containing a cancelable 2- and 3-handle in the sense that the attaching sphere for the 3-handle runs over the 2-handle once. Show that one can do handleslides so that the 2-handle is represented by an unknotted circle disjoint from the rest of the diagram. (Hint: let K be the attaching circle for the 2-handle. The attaching sphere for the 3-handle is an S^2 that intersects the core of the 2-handle in one point. Deleting a neighborhood of that point gives a 2-disk inside the rest of the Kirby diagram. Shrink K along this 2-disk.) Also, give an example where some handleslides are needed.

(0) (Gompf-Stipsicz Exercise 5.1.12(b).) Let L and L' be framed links in $\mathbb{R}^3$ with integral framings so that L' is obtained from L by a handleslide. Prove that L' can be obtained from L by a sequence of blow-ups and blow-downs. (Hint: first consider the case of sliding over a $+1$-framed unknot. Then use the fact that you can reverse crossings by blowing up.) (Remark: Gompf-Stipsicz note that this result is due to Fenn and Rourke, and that it is false if $\mathbb{R}^3$ is replaced by another 3-manifold.)

(0) Let $K \subset S^3$ be the trefoil knot. Find a framed link L inside the solid torus $S^1 \times D^2$ so that performing surgery on L gives $S^3 \setminus \mathrm{nbd}(K)$.

(0) Prove that the disk and upper half plane models for hyperbolic space are isometric. (Hint: a fractional linear transformation sending the disk to the upper half plane will be an isometry.)

(0) *Prove that the isometries of $\mathbb{H}^2$ in the upper half-plane model are exactly the fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. (Hint: You can do this by direct computation. Alternatively, prove that inversion across a circle is an isometry, and that any fractional linear transformation is a composition of inversions. To see these are the only isometries, show the action on unit tangent vectors is transitive.)

(0) Prove that vertical lines and circles perpendicular to the x-axis are geodesics in $\mathbb{H}^2$ in the upper half-plane model and, in fact, these are the only geodesics. (Hint: again, you can do this by direct computation, or using the fact that inversion in a circle is an isometry.)

(0) Recall that an isometry of $\mathbb{H}^2$ is *elliptic* if it has a fixed point in the interior of $\mathbb{H}^2$, *parabolic* if it has one fixed point on the boundary of $\mathbb{H}^2$, and *hyperbolic* if it has two fixed points on the boundary of $\mathbb{H}^2$. Prove that all elliptic isometries are conjugate, all parabolic isometries are conjugate, and all hyperbolic isometries are conjugate. (So, each "looks the same as" a standard model).

(0) Suppose that a finite group G acts on a surface Σ. Given a hyperbolic metric $\langle \cdot, \cdot \rangle$ on Σ, show that

$$\langle\langle v, w \rangle\rangle = \frac{1}{|G|} \sum_{g \in G} \langle dg(v), dg(w) \rangle$$

defines a new hyperbolic metric, and G acts by isometries with respect to this metric.

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