

MATH 636 SPRING 2024
HOMEWORK 8
DUE MAY 28, 2024 (TUESDAY)
 INSTRUCTOR: ROBERT LIPSHITZ

Required problems:

- (0) Hatcher 4.B.2 (p. 428). Note: you solved part of this in a previous homework. You don't have to re-prove that part; just cite it.
- (0) Prove that, for the category of based topological spaces, the reduced suspension functor is left adjoint to the based loop space functor.
- (0) Hatcher 4.2.34 (p. 392).
- (0) Hatcher 4.2.37 (p. 392). (You'll have to read the definitions of Whitehead products and H -spaces first.)

Optional problems:

Some good qual-level problems:

- Hatcher 4.2.14, 4.2.19, 4.2.30.

Some more problems to think about but not turn in:

- In required problem (2), you prove that $\text{Hom}(\Sigma(X, x_0), (Y, y_0)) \cong \text{Hom}((X, x_0), \Omega(Y, y_0))$, naturally. Here, there are two ways you can interpret $\cong$: a bijection of sets, which is easier and what the problem intended, or as a homeomorphism of topological spaces, where the mapping spaces have the compact-open topology. Prove the stronger result, for the category of based locally compact, Hausdorff spaces. In particular, this implies that $[\Sigma(X, x_0), (Y, y_0)] = [(X, x_0), \Omega(Y, y_0)]$ (for X, Y locally compact, Hausdorff).
- In this problem, we show that Hopf's original definition of the Hopf invariant is a homotopy invariant. The problem assumes a little more knowledge of transversality than we have covered in class.
 - (1) Let $f: S^3 \rightarrow S^2$ be a smooth map and p, q, r regular values of f . Show that $\text{lk}(f^{-1}(p), f^{-1}(q)) = \text{lk}(f^{-1}(p), f^{-1}(r))$. (Hint: consider a path γ from q to r with the property that f is transverse to γ .)
 - (2) Show that every continuous map $S^3 \rightarrow S^2$ is homotopic to a smooth map, and that if two smooth maps are homotopic then they are homotopic via a smooth map $S^3 \times [0, 1] \rightarrow S^2$.
 - (3) Now, suppose that $f, g: S^3 \rightarrow S^2$ are homotopic smooth maps and p, q are regular values of both f and g . Show that $\text{lk}(f^{-1}(p), f^{-1}(q)) = \text{lk}(g^{-1}(p), g^{-1}(q))$.
 - (4) Given a continuous map $f: S^3 \rightarrow S^2$, let $\bar{f}$ be a smooth map homotopic to f and p, q regular values of $\bar{f}$. Prove that

$$\text{lk}(\bar{f}^{-1}(p), \bar{f}^{-1}(q))$$

is an invariant of the homotopic class of f .

- Let $K_1, K_2 \subset S^3$ be knots (smoothly embedded circles) and $\Sigma_1, \Sigma_2 \subset B^4$ smoothly embedded, orientable surfaces with $\partial \Sigma_i = K_i$. Show that $\text{lk}(K_1, K_2) = \pm \# \Sigma_1 \cap \Sigma_2$. (Optionally, note that orientations of the K_i induce orientations of the Σ_i , and the sign in the formula becomes +.)

Email address: `lipshitz@uoregon.edu`