

MATH 636 SPRING 2024
HOMEWORK 6
DUE MAY 13, 2024
 INSTRUCTOR: ROBERT LIPSHITZ

Required problems:

- (1) Hatcher 4.2.9 (p. 389). (For the second part, I found it easier to ignore Hatcher's hint.)
- (2) Hatcher 4.2.12 (p. 389).
- (3) Hatcher 4.2.16 (p. 389).
- (4) Hatcher 4.2.18 (p. 389).

Optional problems:

Some good qual-level problems:

- Hatcher 4.2.2, 4.2.8, 4.2.13, 4.2.15 (p. 389).

Some more problems to think about but not turn in:

- Homology with local coefficients might be helpful context for understanding the orientation cover M_R of a manifold M . Read Hatcher's Section 3.H and try problems 3.H.1 and 3.H.2.
- This problem assumes you know the basics of Čech cohomology of sheaves. A *good cover* of a space X is a cover by open sets so that every finite intersection of sets in the cover is either empty or contractible. Given a bundle of groups $p: E \rightarrow X$, define a sheaf $\mathcal{F}$ with $\mathcal{F}(U)$ the group of sections of E over U (i.e., continuous maps $s: U \rightarrow E$ with $p \circ s = \mathbb{1}_U$). The goal of this exercise is to prove that if X admits a finite good cover $\mathcal{U} = \{U_1, \dots, U_k\}$ then the local cohomology of X with coefficients in E is the Čech cohomology of $\mathcal{F}$. (The condition that the good cover be finite is not needed, but the condition that X admits a good cover is.)
 - (1) Prove that $\mathcal{F}$ is a sheaf.
 - (2) Show that if U is contractible then $E|_U$ is a trivial bundle of groups.
 - (3) Generalize the proof of the Mayer-Vietoris sequence to show that for each n there is an exact sequence

$$0 \rightarrow C_{\mathcal{U}}^n(X; E) \rightarrow \bigoplus_i C^n(U_i; E|_{U_i}) \rightarrow \bigoplus_{i < j} C^n(U_i \cap U_j; E|_{U_i \cap U_j}) \rightarrow \bigoplus_{i < j < \ell} C^n(U_i \cap U_j \cap U_\ell; E|_{U_i \cap U_j \cap U_\ell}) \rightarrow \dots$$

- (4) Show that the maps in the exact sequence above are chain maps, with respect to the differential $\delta: C_{\mathcal{U}}^n(X; E) \rightarrow C_{\mathcal{U}}^{n+1}(X; E)$ and $\delta: C^n(U_{i_1} \cap \dots \cap U_{i_j}; E|_{U_{i_1} \cap \dots \cap U_{i_j}}) \rightarrow C^{n+1}(U_{i_1} \cap \dots \cap U_{i_j}; E|_{U_{i_1} \cap \dots \cap U_{i_j}})$.
- (5) Now, view

$$\bigoplus_i C^n(U_i; E|_{U_i}) \rightarrow \bigoplus_{i < j} C^n(U_i \cap U_j; E|_{U_i \cap U_j}) \rightarrow \bigoplus_{i < j < \ell} C^n(U_i \cap U_j \cap U_\ell; E|_{U_i \cap U_j \cap U_\ell}) \rightarrow \dots$$

as a bicomplex, where the horizontal differential is the maps in the exact sequence and the vertical differential is the differential on the singular cochain complex (with local coefficients). Let D^* be the associated total complex. Show that there is a chain map $C_{\mathcal{U}}^n(X; E) \rightarrow D^*$ inducing an isomorphism on homology.

- (6) Show that the homology of $C^*(U_{i_1} \cap \cdots \cap U_{i_j}; E|_{U_{i_1} \cap \cdots \cap U_{i_j}})$ (with respect to the differential on the singular cochain complex) is:
- 0 if $U_{i_1} \cap \cdots \cap U_{i_j} = \emptyset$.
 - 0 in gradings $* > 0$,
 - G in grading 0 if $U_{i_1} \cap \cdots \cap U_{i_j} \neq \emptyset$.
- (7) Prove that the homology of D^* is the Čech cohomology of the sheaf $\mathcal{F}$. Since this is isomorphic to $H^n(X; E)$, this proves the result.

Email address: lipshitz@uoregon.edu