

MATH 636 SPRING 2024
HOMEWORK 3
DUE APRIL 22, 2024
 INSTRUCTOR: ROBERT LIPSHITZ

Required problems:

- (1) Hatcher 3.3.16 (p. 259).
- (2) Hatcher 3.3.24 (p. 259).
- (3) Hatcher 3.3.30 (p. 260).
- (4) Hatcher 3.3.33 (p. 260).

Optional problems:

Some good qual-level problems:

- Hatcher 3.3.20, 3.3.21 (minus the parenthetical question), 3.3.25, 3.3.26.

Some more problems to think about but not turn in:

- Hatcher 3.3.22, 3.3.23, 3.3.27–29.
- Let X be a topological space. A *locally-finite singular n -chain* in X is a (possibly infinite) formal linear combination $\sum k_\sigma \sigma \in \prod_{\sigma: \Delta^n \rightarrow X} \mathbb{Z}$ so that for each compact set $C \subset X$,

$$\{\sigma \mid k_\sigma \neq 0 \text{ and } \sigma^{-1}(C) \neq \emptyset\}$$

is finite. Let $C_n^{BM}(X)$ denote the abelian group of locally-finite singular n -chains in X .

- (1) Show that the “obvious” boundary operator $\partial: C_n^{BM}(X) \rightarrow C_{n-1}^{BM}(X)$,

$$\partial \sum_{\sigma} k_{\sigma} \sigma = \sum_{\sigma} \sum_{i=0}^n (-1)^i k_{\sigma} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

is well-defined and makes $C_n^{BM}(X)$ into a chain complex. (*Hint.* This would *not* be true without the “locally-finite” condition, so your proof had better use it!)

- (2) The homology of (C_*^{BM}, ∂) is called the *Borel-Moore homology of M* , and denoted $H_*^{BM}(X)$. Observe that if X is compact then $H_*^{BM}(X) \cong H_*(X)$.

Compute $H_*^{BM}(\mathbb{R})$.

- (3) Show that if M is a connected, orientable n -manifold, not necessarily compact, then $H_n^{BM}(M) \cong \mathbb{Z}$. In particular, any oriented n -manifold M has a fundamental class $[M] \in H_n^{BM}(M)$.
- (4) Show that there is a well-defined cap product

$$H_{i+j}^{BM}(X) \otimes H^i(X) \rightarrow H_j^{BM}(X).$$

- (5) Imitate Hatcher’s proof of Poincaré duality to show that for any oriented n -manifold M ,

$$[M] \cap \cdot: H^i(X) \rightarrow H_{n-i}^{BM}(X)$$

is an isomorphism. In particular, this gives an alternative proof of Poincaré duality for compact manifolds.

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