

MATH 636 SPRING 2024
HOMEWORK 10
DUE JUNE 8, 2024
INSTRUCTOR: ROBERT LIPSHITZ

Required problems:

- (0) Hatcher 4.3.16 (p. 420).
- (0) Up to homotopy equivalence, how many CW complexes X are there with $\pi_2(X) \cong \mathbb{Z}$, $\pi_3(X) \cong \mathbb{Z}/2\mathbb{Z}$, and all other homotopy groups trivial? Describe each of these spaces as explicitly as you can.
- (0) Fix $n \geq 2$. Describe explicitly the first $2n$ stages of a Postnikov tower for $\mathbb{C}P^n$ consisting of principal fibrations (up through what Hatcher labels X_n), and compute the corresponding k -invariants $k_1, \dots, k_{2n-1}$. Show further that k_{2n} is nontrivial.

Optional problems:

Some good qual-level problems:

- Hatcher 4.3.11, 4.3.12, 4.3.15, 4.3.17, 4.3.20, 4.3.21, 4.3.22.

Some more problems to think about but not turn in:

- (1) Fix $n \geq 2$. Let X be a CW complex such that $\pi_i(X) = 0$ for $i < n$. Show that there is a map $X \rightarrow K(\pi_n(X), n)$ inducing an isomorphism on π_n .
- (2) Given a simply-connected space X and an integer n , by taking iterated homotopy fibers, build another space Y and a map $f: Y \rightarrow X$ so that $\pi_i(Y) = 0$ for $i < n$ and $f_*: \pi_i(Y) \rightarrow \pi_i(X)$ is an isomorphism for $i \geq n$. (Do not use the existence of Moore-Postnikov towers.)
- Hatcher 4.3.8, 4.3.9.
- Deduce the bundle homotopy lemma for fibrations as stated in class from Hatcher, Proposition 4.62.
- The point of this exercise is to illustrate that inverse limits can be somewhat bonkers, even for nice maps between nice spaces. Consider the inverse system

$$S^1 \leftarrow S^1 \leftarrow S^1 \leftarrow \dots,$$

where each map in the sequence is the double cover $z \mapsto z^2$. Let S_n^1 denote the n^{th} term in this sequence and $p_n: S_{n+1}^1 \rightarrow S_n^1$ the n^{th} map. Let $S_\infty^1 = \varprojlim S_n^1$.

- (1) Let $(z_1, z_2, \dots) \in S_\infty^1$. Show that the universal covering maps $q_n: \mathbb{R} \rightarrow S_n^1$, $q_n(t) = z_n^{-1} e^{2\pi i t/2^n}$, induce a continuous map $q_\infty: \mathbb{R} \rightarrow S_\infty^1$.
- (2) Show that the image of q_∞ is a path component of S_∞^1 .
- (3) Show that q_∞ is not surjective.
- (4) Show that the image of q_∞ is dense.
- (5) Conclude that S_∞^1 is not locally path connected.

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