

MATH 636 SPRING 2024

HOMEWORK 1

DUE APRIL 8, 2024

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Required problems:

- (1) Use the Künneth theorem to compute the homology groups of the following spaces. Then use the universal coefficient theorem to compute their cohomology groups:
 - (a) $S^1 \times M_g$, where M_g is the closed, orientable surface of genus g .
 - (b) $K \times K$, where K is the Klein bottle.
 - (c) $M(\mathbb{Z}/p\mathbb{Z}, m) \times M(\mathbb{Z}/q\mathbb{Z}, n)$ (a product of two Moore spaces), where p and q are (not necessarily distinct) primes.
 - (d) $M(\mathbb{Z}/4\mathbb{Z}, 2) \times M(\mathbb{Z}/6\mathbb{Z}, 3)$.
- (2) Hatcher 3.2.15 (p. 230). (You can skip the “and the spaces in the preceding three exercises” part.)
- (3) Hatcher 3.B.3 (p. 280).

Optional problems:

Some good qual-level problems:

- (4) Some good qual-level problems: Hatcher 3.2.16, 3.2.18, 3.B.1.

Some more problems to think about but not turn in:

- (5) Let $F, G: C_*(X) \otimes C_*(Y) \rightarrow C_*(X) \otimes C_*(Y)$ be chain maps, for each pair of spaces X, Y , with the following properties:
 - (a) F and G are natural, in the sense that given $f: X \rightarrow X'$, $g: Y \rightarrow Y'$,

$$F(f_*(\alpha) \otimes g_*(\beta)) = (f_* \otimes g_*)(F(\alpha \otimes \beta))$$

$$G(f_*(\alpha) \otimes g_*(\beta)) = (f_* \otimes g_*)(G(\alpha \otimes \beta)).$$

- (b) On $C_0(X) \otimes C_0(Y)$, F and G are the identity map.

Prove that F is chain homotopic to G . (This was one of the steps in our brief sketch of the Eilenberg-Zilber theorem.)

- (6) By the Eilenberg-Zilber theorem, $C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$; let $E: C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y)$ be one of the chain homotopy equivalences constructed in the proof of the Eilenberg-Zilber theorem. There is an induced chain homotopy equivalence

$$E^T: C^*(X) \otimes C^*(Y) \rightarrow C^*(X \times Y).$$

Now, define a map $\cup: C^i(X) \otimes C^j(X) \rightarrow C^{i+j}(X)$ to be the composition

$$C^i(X) \otimes C^j(X) \xrightarrow{E^T} C^{i+j}(X \times X) \xrightarrow{\Delta^*} C^{i+j}(X),$$

where $\Delta: X \rightarrow X \times X$ is the diagonal map.

- (a) Show that $\cup$ induces a map $\cup: H^i(X) \otimes H^j(X) \rightarrow H^{i+j}(X)$.

- (b) Show that the map $\cup$ on homology is independent of the choice of chain homotopy equivalence E .
- (c) Show that the this cup product on cohomology is natural, unital, and associative. (Associativity probably takes a little work.)
- (d) Show that this definition of the cup product agrees with the one given in Hatcher.

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