

MATH 636 HOMEWORK 6
DUE MAY 7, 2021.

INSTRUCTOR: ROBERT LIPSHITZ

- (1) Hatcher 4.2.22 (p. 390).
- (2) The case $n > 1$ of Hatcher 4.2.23 (p. 390). (You can view the $n = 1$ case as an optional exercise.)
- (3) Hatcher 4.2.31 (p. 392). (You may assume the nullhomotopy of the inclusion $F \hookrightarrow E$ is constant on $\tilde{x}_0$.)
- (4) Hatcher 4.B.2 (p. 428). Note: you solved part of this in a previous homework. You don't have to re-prove that part; just cite it.

Suggested review / qualifying exam practice (not to turn in):

- (1) 4.2.28, 4.2.29, 4.2.34, 4.B.1.
- (2) Give a direct proof of exactness for the long exact sequence

$$\cdots \rightarrow \pi_n(F, \tilde{x}_0) \rightarrow \pi_n(E, \tilde{x}_0) \rightarrow \pi_n(B, x_0) \rightarrow \pi_{n-1}(F, \tilde{x}_0) \rightarrow \cdots$$

associated to a fibration $F \rightarrow E \rightarrow B$. (That is, don't use exactness of the long exact sequence for a pair.)

More problems to think about but not turn in:

- (1) In this problem, we show that Hopf's original definition of the Hopf invariant is a homotopy invariant. The problem assumes a little more knowledge of transversality than we have covered in class.
 - (a) Let $f: S^3 \rightarrow S^2$ be a smooth map and p, q, r regular values of f . Show that $\text{lk}(f^{-1}(p), f^{-1}(q)) = \text{lk}(f^{-1}(p), f^{-1}(r))$. (Hint: consider a path γ from q to r with the property that f is transverse to γ .)
 - (b) Show that every continuous map $S^3 \rightarrow S^2$ is homotopic to a smooth map, and that if two smooth maps are homotopic then they are homotopic via a smooth map $S^3 \times [0, 1] \rightarrow S^2$.
 - (c) Now, suppose that $f, g: S^3 \rightarrow S^2$ are homotopic smooth maps and p, q are regular values of both f and g . Show that $\text{lk}(f^{-1}(p), f^{-1}(q)) = \text{lk}(g^{-1}(p), g^{-1}(q))$.
 - (d) Given a continuous map $f: S^3 \rightarrow S^2$, let $\bar{f}$ be a smooth map homotopic to f and p, q regular values of $\bar{f}$. Prove that

$$\text{lk}(\bar{f}^{-1}(p), \bar{f}^{-1}(q))$$

is an invariant of the homotopic class of f .

- (2) Let $K_1, K_2 \subset S^3$ be knots (smoothly embedded circles) and $\Sigma_1, \Sigma_2 \subset B^4$ smoothly embedded, orientable surfaces with $\partial \Sigma_i = K_i$. Show that $\text{lk}(K_1, K_2) = \pm \# \Sigma_1 \cap \Sigma_2$. (Optionally, note that orientations of the K_i induce orientations of the Σ_i , and the sign in the formula becomes +.)

- (3) Let $M^{n-1}, N^{n-1} \subset S^{2n-1}$ be closed, orientable submanifolds. Suppose that $M = \partial P$ for some n -dimensional manifold-with-boundary $P \subset S^{2n-1}$. Show that if $N \not\subset P$ then $\text{lk}(M, N) = \pm \#P \cap N$.

Email address: lipshitz@uoregon.edu