MATH 636 HOMEWORK 2
DUE APRIL 9, 2021.

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(1) Given a fiber bundle $p\colon E \to B$ with fiber a sphere S^n, satisfying an orientability condition (see Optional Problem 1), there is a long exact sequence
$$\cdots \to H^k(B) \longrightarrow H^{n+k+1}(B) \longrightarrow H^{n+k+1}(E) \longrightarrow H^{k+1}(B) \to \cdots,$$
called the Gysin sequence. Here, the map $H^{n+k+1}(B) \to H^{n+k+1}(E)$ is p^* and the map $H^k(B) \to H^{n+k+1}(B)$ is the cup product with a particular cohomology class e.

Suppose that E is the unit sphere bundle in a vector bundle E', $E = SE'$. Suppose further that E' is orientable, so the Thom isomorphism theorem holds. Deduce the existence of the Gysin sequence from the Thom isomorphism theorem. (In the process, you should also describe the cohomology class e.)

(2) Show that S^{2n+1} is a fiber bundle over $\mathbb{C}P^n$ with fiber S^1, and in fact this is the sphere bundle associated to an orientable vector bundle. (You do not have to give a detailed proof that the vector bundle you write down is a vector bundle, or orientable.) Then use the Gysin sequence to compute the cohomology ring of $\mathbb{C}P^n$.

(3) Given an n-dimensional vector bundle $p\colon E \to B$, let $e(E) \in H^n(B)$ be the cohomology class e from Problem 1 associated to the bundle $SE \to B$. Show that if E has a nonvanishing section (or equivalently SE has a section) then $e(E) = 0$. (Hint: this is easy.) Deduce that the vector bundle from Problem 2 does not have a nonvanishing section.

(4) What does the existence of the Gysin sequence imply about when there are fiber bundles $S^m \to S^n$ with fiber a sphere S^k? (i.e., what relation does it imply among m, n, k?)

Suggested review / qualifying exam practice (not to turn in):

(1) Prove the Thom isomorphism theorem.

(2) Let STS^2 be the unit sphere bundle of the tangent bundle of S^2. Use the Mayer-Vietoris sequence to compute the cohomology of STS^2, and deduce that TS^2 has no nonvanishing section.

More problems to think about but not turn in:

(1) Call a sphere bundle $p\colon E \to B$ (with fiber S^n) *orientable* if for every loop $\gamma\colon S^1 \to B$, $\gamma^*E \cong S^1 \times S^n$ (as bundles over S^1). Prove that if $E \to B$ is an orientable sphere bundle then the Thom isomorphism holds for the bundle $DE \to B$ where DE is the mapping cone of the projection $p\colon E \to B$ (so DE is a bundle with fibers D^{n+1}). Deduce that the Gysin exists for any orientable sphere bundle.

(2) Give an example of a non-orientable sphere bundle for which the Gysin sequence does not exist.

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