

MATH 636 HOMEWORK 1
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Updated: typo fixed in (4b).

- (1) Hatcher 3.3.24 (p. 259). Deduce also that if M is simply connected then $H_i(M) \cong H_i(S^3)$ for each i .
- (2) Hatcher 3.3.31 (p. 260).
- (3) Hatcher 3.3.32 (p. 260).
- (4) Let X be a closed, orientable $4n$ -dimensional manifold.
 - (a) Recall that the *signature* of a bilinear form is the number of positive eigenvalues minus the number of negative eigenvalues. Prove the following algebraic lemma.
Lemma. Let V be a $2k$ -dimensional $\mathbb{R}$ -vector space and $g: V \times V \rightarrow \mathbb{R}$ a nondegenerate, symmetric, bilinear form. If there is a k -dimensional subspace $W \subset V$ so that $g(v, w) = 0$ for all $v, w \in W$ then the signature of g is zero.
 - (b) Fix an orientation $[X]$ for X . Define the *signature* $\sigma(X)$ of X to be the signature of the bilinear form

$$H^{2n}(X; \mathbb{R}) \times H^{2n}(X; \mathbb{R}) \rightarrow \mathbb{R}$$

$$(a, b) \mapsto \langle a \cup b, [X] \rangle.$$

Show that if X is the boundary of a compact, orientable $(4n + 1)$ -dimensional manifold-with-boundary then $\sigma(X) = 0$.

- (c) Deduce that $\mathbb{C}P^{2n}$ is not the boundary of an orientable $(4n + 1)$ -dimensional manifold.

Suggested review / qualifying exam practice (not to turn in):

- (1) Use Lefschetz duality to show that the cup product

$$H^k(D^k \times D^\ell, S^{k-1} \times D^\ell) \otimes H^\ell(D^k \times D^\ell, D^k \times S^{\ell-1}) \xrightarrow{\cup} H^{k+\ell}(D^k \times D^\ell, \partial(D^k \times D^\ell))$$

is an isomorphism. (This is a lemma we proved earlier, by more direct techniques.)

- (2) A *2-component link* is a pair of disjoint, smooth embeddings $K_1, K_2: S^1 \hookrightarrow S^3$. Given a 2-component link, the *linking number* of L is defined as follows:

- (a) By Alexander duality, $H_1(S^3 \setminus K_1(S^1)) \cong H^1(K_1(S^1)) \cong H^1(S^1) \cong \mathbb{Z}$, where the last isomorphism sends the positive generator of $H^1(K_1) = H^1(S^1)$ to 1. On the other hand, $(K_2)_*([S^1])$ is an element of $H_1(S^3 \setminus K_1(S^1))$. The image of this element in $\mathbb{Z}$ is the linking number of the link.
- (b) Alternatively, let $\text{nbdd}(K_1(S^1))$ be a small solid torus neighborhood of $K_1(S^1)$. Then $H_2(S^3 \setminus \text{nbdd}(K_1(S^1)), \partial \text{nbdd}(K_1(S^1))) \rightarrow H_1(\partial \text{nbdd}(K_1(S^1))) \rightarrow H_1(K_1(S^1))$ is an isomorphism (this needs proof), so the generator $(K_1)_*([S^1])$ gives a preferred generator of $H_2(S^3 \setminus \text{nbdd}(K_1(S^1)), \partial \text{nbdd}(K_1(S^1)))$. By Lefschetz duality, $H_2(S^3 \setminus \text{nbdd}(K_1(S^1)), \partial \text{nbdd}(K_1(S^1))) \cong H^1(S^3 \setminus \text{nbdd}(K_1))$. Evaluating this preferred generator on $(K_2)_*[S^1]$ gives an integer, the linking number of the link.

Show that these two definitions are equivalent.

More problems to think about but not turn in:

- (1) Let Y be a compact, orientable 3-manifold-with-boundary, and $F = \partial Y$, a closed surface of genus g .
 (a) Fix an orientation (fundamental class) $[F]$ for F . Define a pairing $\omega: H^1(F) \times H^1(F) \rightarrow \mathbb{Z}$ by

$$\omega(a, b) = \langle a \cup b, [F] \rangle = (a \cup b) \cap [F].$$

Prove that ω is skew-symmetric and non-degenerate.

- (b) Let $K \subset H^1(F)$ be the image of $j^*: H^1(Y) \rightarrow H^1(F)$ (where $j: F \rightarrow Y$ is inclusion). Show that for any $a, b \in K$, $\omega(a, b) = 0$.
 (c) Deduce from Hatcher, problem 3.3.28, that the rank of K is at most g . In particular, deduce that if $g > 0$ then Y is not simply connected.
 (d) Show that the rank of K is exactly g . (In other words, K is a Lagrangian subspace of $(H^1(F), \omega)$.)
 (2) Hatcher 3.3.35.
 (3) Read through Hatcher's first computation of the ring structure on $H^*(\mathbb{C}P^n)$ with Poincaré duality in mind, and see how he came up with it.
 (4) Let Y be a closed, smooth, oriented 3-manifold. Show that every homology class $\alpha \in H_2(Y; \mathbb{Z})$ is represented by some embedded surface $\Sigma \subset Y$, as follows. Let $b = \text{PD}(\alpha) \in H^1(Y; \mathbb{Z})$. From last quarter, there is a map $f: Y \rightarrow S^1$ so that $f^*(d\theta) = b$, where $d\theta \in H^1(S^1; \mathbb{Z})$ is the preferred generator. (Here, perhaps we are using the fact that Y is homeomorphic to a CW complex.) Show that f is homotopic to a smooth map $g: Y \rightarrow S^1$. Let $p \in S^1$ be a regular value of g . Show that $\Sigma = g^{-1}(p)$ represents α .
 (5) Let $p(z_0, z_1, z_2), q(z_0, z_1, z_2) \in \mathbb{C}[z_0, z_1, z_2]$ be degree d homogeneous polynomials so that, viewing p and q as maps $\mathbb{R}^6 \cong \mathbb{C}^3 \rightarrow \mathbb{C} \cong \mathbb{R}^2$, 0 is a regular value of p and q .
 (a) Show that $\{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid p(z_0, z_1, z_2) = 0\}$ is a smooth surface Σ_p .
 (b) Show that there is a smooth path p_t of homogeneous polynomials so that $p_0 = p$ and $p_1 = q$, and 0 is a regular value of p_t for all t . (Hint: show that 0 not being a regular value is a polynomial equation on the coefficients of p_t , and hence has real codimension 2.)
 (c) Show that Σ_p and Σ_q represent the same element of $H_2(\mathbb{C}P^2)$, as follows. Let p_t be as in the previous part. Then

$$\{(t, [z_0 : z_1 : z_2]) \in [0, 1] \times \mathbb{C}P^2 \mid p_t(z_0, z_1, z_2) = 0\}$$

is an orientable 3-manifold inside $[0, 1] \times \mathbb{C}P^2$, with boundary $-\Sigma_p \amalg \Sigma_q$ (where the $-$ denotes orientation reversal). Hence, from an exercise last week, $[\Sigma_p] = [\Sigma_q] \in H_2([0, 1] \times \mathbb{C}P^2) = H_2(\mathbb{C}P^2)$.

- (6) Given $n \in \mathbb{N}$, let $Y_n = \{z \in \mathbb{C} \mid z^n \in \mathbb{R}_{\geq 0}\}$, so Y_n looks like a graph with one central vertex and n spokes. A *branched surface* is a topological space X so that each point $x \in X$ has a neighborhood $U \ni x$ homeomorphic to either $\mathbb{R}^2$ or $(0, 1) \times Y_n$ for some n . (That is, near each point Y either looks like $\mathbb{R}^2$ or like n half-planes coming together.) A *p-branched surface* is a branched surface where the only Y_n s which occur have n divisible by p . (So, for instance, a 2-branched surface can look locally like a

plane, like 4 half-planes coming together, like 6 half-planes coming together, and so on.)

- (a) Suppose X is a compact 2-branched surface. Show that X has a fundamental class $[X] \in H_2(X; \mathbb{F}_2)$.
- (b)
- (c) (Harder) Read what the Bockstein homomorphism is and relate it to representing mod-2 homology classes by 2-branched surfaces.
- (d) (Harder) Generalize the previous two parts to p -branched surfaces for p an odd prime. Here, you will need a notion of orientability for p -branched surfaces.

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