MATH 431/531 FALL 2021
HOMEWORK 5
DUE OCTOBER 27, 2021.

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Problems:

(1) Lee, Exercise 2.42.
(2) Lee, Exercise 2.51.
(3) Lee, Problem (not exercise) 2.18.
(4) Lee, Problem (not exercise) 2.20.
(5) Lee, Problem (not exercise) 2.22.

Challenge problems (required for Math 531, optional for 431):

(6) Let $C^0[0,1]$ denote the set of continuous functions $f\colon [0,1] \to \mathbb{R}$. More generally, for $k \geq 0$ let $C^k[0,1]$ denote the set of functions $f\colon [0,1] \to \mathbb{R}$ so that f and its first k derivatives (i.e., f, f', f'', $f^{(3)}$,...,$f^{(k)}$) are continuous. Define a metric d_0 on $C^0[0,1]$ by

$$d_0(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\}.$$

(We will prove in a few weeks that this supremum is always finite; you may assume that for now.) Define a metric d_k on $C^k[0,1]$ by

$$d_k(f,g) = d_0(f,g) + d_0(f',g') + \cdots + d_0(f^{(k)}, g^{(k)}).$$

(a) Prove that $(C^k[0,1], d_k)$ is a metric space.

(b) Prove that the topology on $C^k[0,1]$ induced by d_k is separable.

You may use the following fact, which I will ask you to prove later in the quarter. A *piecewise-linear function* $g\colon [0,1] \to \mathbb{R}$ is a function defined as follows. Fix some $n \in \mathbb{N}$, $0 = a_0 < a_1 < \cdots < a_n = 1$, $b_0 \in \mathbb{R}$, and $m_i \in \mathbb{R}$, $i = 0, \dots, n-1$. For $i = 1, \dots, n-1$, define b_i inductively by $b_{i+1} = b_i + m_i(a_{i+1} - a_i)$. Then the function $g = g_{a_0,\dots,a_n;b_0;m_0,\dots,m_{n-1}}$ defined by

$$g(x) = b_i + m_i(x - a_i) \qquad \text{if } a_i \leq x \leq a_{i+1}$$

is piecewise-linear. (i.e., the piecewise-linear functions are the ones that can be written in this form.)

Fact (which you may assume). Given $f \in C^0[0,1]$ and $\epsilon > 0$ then there is $n \in \mathbb{N}$ and $a_0, \dots, a_n, b_0, m_0, \dots, m_{n-1} \in \mathbb{Q}$ (with $0 = a_0 < a_1 < \cdots < a_n = 1$) so that $d_0(f, g_{a_0,\dots,a_n;b_0;m_0,\dots,m_{n-1}}) < \epsilon$.

(Hint: integrate repeatedly.)

(c) Deduce that the topology on $C^k[0,1]$ induced by d_k is second countable.

(d) Let $C^\infty[0,1] = \bigcap_k C^k$ denote the set of functions $f\colon [0,1] \to \mathbb{R}$ so that $f^{(k)}$ is continuous for all $k \geq 0$. Consider the basis

$$\mathcal{B} = \{U \cap C^\infty[0,1] \mid U \subset C^k[0,1] \text{ open for some } k \geq 0\}.$$

That is, a set is in $\mathcal{B}$ if it is the intersection of an open set in $C^k[0,1]$, for some k, with $C^\infty[0,1]$. Prove that this is a basis for a topology on $C^\infty[0,1]$. The corresponding topology is called the C^∞ *topology*.

(e) Is $C^\infty[0,1]$ second countable? Separable? First countable? Justify your answers.

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