MATH 431/531 FALL 2021
HOMEWORK 3
DUE OCTOBER 13, 2021.

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Problems:

(1) Prove parts (b,d,g) of Proposition 2.8 in Lee.

(2) Let X be a topological space and A a subset of X. There is a map $i\colon A \to X$ given by $i(a) = a$; i is called the *inclusion map*.

  (a) Prove that if we give A the subspace topology then the inclusion map is continuous.

  (b) Given topologies $\mathcal{U}$, $\mathcal{V}$ on A, we say that $\mathcal{V}$ is *finer* than $\mathcal{U}$ if $\mathcal{U} \subset \mathcal{V}$, i.e., if every set which is open with respect to $\mathcal{U}$ is also open with respect to $\mathcal{V}$.

  Suppose that $\mathcal{V}$ is a topology on A so that the inclusion map is continuous with respect to $\mathcal{V}$. Prove that $\mathcal{V}$ is finer than the subspace topology. (In other words, the subspace topology is the coarsest topology for which the inclusion map is continuous.)

(3) Let X be a set. What subsets of X are dense in X with respect to the discrete topology? With respect to the indiscrete topology?

(4) Let $A = (-7, 0) \cup \{1/n \mid n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$, with the usual (metric) topology. (Here, $\mathbb{Z}_{>0}$ denotes the positive integers. Find:

  (a) The interior of A.

  (b) The exterior of A.

  (c) The closure of A.

  (d) The boundary of A.

  (e) The limit points of A.

  (f) The isolated points of A.

No justification needed in this problem; just the answers.

(5) Consider the unit circle $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$. What is the boundary of the subspace A? (Justification optional.)

(6) Find a function $f\colon \mathbb{R} \to \mathbb{R}$ (both with the standard topology) which is

  (a) Discontinuous at every point.

  (b) Continuous at exactly one point.

  (c) Continuous exactly at $\mathbb{Z} \subset \mathbb{R}$.

(Here, we are using the metric space definition of continuity at a point.)

Challenge problems (required for Math 531, optional for 431):

(7) (a) Solve Lee Exercise 2.6.

  (b) Using the previous part, show that if X is a topological space and A is a subspace of X then there is a unique coarsest topology on A so that the inclusion map $i\colon A \to X$ is continuous. Do *not* use the explicit description of the subspace topology.

(8) Consider $(0,1)^2$ with the lexicographic order, i.e.,

$$(a,b) < (c,d) \equiv \begin{cases} a < c & \text{or} \\ a = c \ \text{ and } b < d \end{cases}.$$

Let $\mathcal{U}$ be the order topology on $(0,1)^2$ corresponding to this order $<$. Find a metric on $(0,1)^2$ so that the induced topology is $\mathcal{U}$. (i.e., show that this topology is *metrizable*).

Bonus problems (not required for anyone):

(9) Consider $[0,1]^2$ with the lexicographic order. Show that the order topology on $[0,1]^2$ is *not* metrizable (i.e., is not induced by any metric).

(10) Let X be a topological space and A a subset of X. The *Cantor derivative* of A is the set of limit points of A.

- (a) Find a nonempty set $A \subset \mathbb{R}$ whose first Cantor derivative is empty.
- (b) Find a set $A \subset \mathbb{R}$ whose first Cantor derivative is non-empty but whose second Cantor derivative is empty.
- (c) For any positive integer n, find a set $A \subset \mathbb{R}$ whose $(n-1)^{\text{st}}$ Cantor derivative is non-empty but whose n^{th} Cantor derivative is empty.
- (d) The ω^{th} Cantor derivative of A is

$$A^{(\omega)} = A \cap A' \cap A'' \cap A''' \cap \dots$$

  (where I am using primes to denote Cantor derivatives). Find a subset $A \subset \mathbb{R}$ so that all finite Cantor derivatives of A are non-empty but $A^{(\omega)}$ is empty.
- (e) Given any countable ordinal α define the α^{th} Cantor derivative of A. (Hint: there are two cases, depending on whether α is a successor ordinal or a limit ordinal.) Find a subset $A \subset \mathbb{R}$ so that if $\beta < \alpha$ then $A^{(\beta)} \neq$ but $A^{\alpha} = \emptyset$.
- (f) A closed subset $A \subset \mathbb{R}$ is *perfect* if $A = A'$. Show that for any closed subset $A \subset \mathbb{R}$ there is a countable ordinal α so that $A^{(\alpha)}$ is perfect.

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