MATH 431/531 FALL 2021
HOMEWORK 2
DUE OCTOBER 6, 2021.

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Problems:

(1) Suppose that d is a metric on X, and d_{dis} is the discrete metric on X. Prove that the identity map $Id\colon (X, d_{dis}) \to (X, d)$, $Id(x) = x$, is continuous with respect to d_{dis} and d.

(2) Consider the standard metric d_{std} on $\mathbb{R}$, $d_{std}(x, y) = |x - y|$, and the discrete metric d_{dis} on $\mathbb{R}$. Prove that the identity map $Id\colon (\mathbb{R}, d_{std}) \to (\mathbb{R}, d_{dis})$ is not continuous.

(3) Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces. Give $X \times Y$ the product metric. Suppose that $f\colon Z \to X$ and $g\colon Z \to Y$ are continuous functions. Prove that $h\colon Z \to X \times Y$, $h(z) = (f(z), g(z))$ is a continuous function.

(4) Suppose that (X, d_X) and (Y, d_Y) are metric spaces, and let $y \in Y$. Prove that the constant function $f\colon X \to Y$ given by $f(x) = y$ for all $x \in X$ is continuous.

(5) In this problem, *continuous* always means with respect to the standard metric on $\mathbb{R}^n$.
   (a) Prove that the map $a\colon \mathbb{R}^2 \to \mathbb{R}$ given by $a(x, y) = x + y$ is continuous.
   (b) Prove that the map $m\colon \mathbb{R}^2 \to \mathbb{R}$ given by $m(x, y) = xy$ is continuous.
   (c) Prove that the function $f(x)\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 9x^2 + 3x - 2$ is continuous. Hint: use the previous two parts and the previous two problems.

Challenge problems (required for Math 531, optional for 431):

(6) Generalizing Problem (5c), prove that if $p(x)$ is any polynomial, viewed as a map $\mathbb{R} \to \mathbb{R}$, then $p(x)$ is continuous.

(7) Consider the standard metric d_{std} on $\mathbb{R}^2$ and also the metric d on $\mathbb{R}^2$ given by $d((x, y), (x', y')) = \max\{|x - x'|,\ |y - y'|\}$.
   (a) Prove that d is a metric.
   (b) Describe in words $B_r(p; d) = \{q \in \mathbb{R}^2 \mid d(p, q) < r\}$, the ball of radius r centered at p with respect to d.
   (c) Let $B_r(p; d_{std}) = \{q \in \mathbb{R}^2 \mid d_{std}(p, q) < r\}$. Prove that $B_r(p; d_{std})$ is open with respect to d.
   (d) Prove that $B_r(p; d)$ is open with respect to d_{std}.
   (e) Prove that a set U in $\mathbb{R}^2$ is open with respect to d_{std} if and only if U is open with respect to d.

(f) Suppose that (X, d_X) is a metric space and $f\colon \mathbb{R}^2 \to X$ is a function. Prove that f is continuous with respect to d if and only if f is continuous with respect to d_{std}. (Hint: use the previous part.)

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