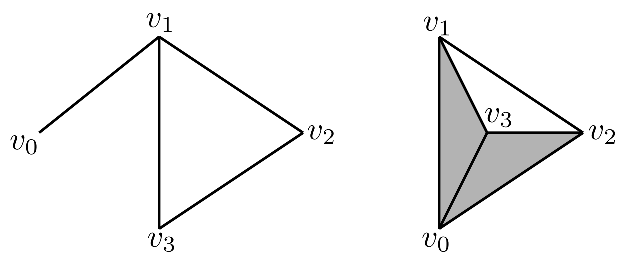
MATH 634 HOMEWORK 8
DUE NOVEMBER 14, 2018.

INSTRUCTOR: ROBERT LIPSHITZ

Required problems, to turn in:

(1) Give abstract simplicial complexes whose geometric realizations are the following simplicial complexes. Compute the simplicial homology and cohomology groups of these complexes. (Note: the shaded triangles are 2-simplices.)



(a) (b)

(2) Recall that $\delta = \delta_n : C^n(X) \to C^{n+1}(X)$ is the coboundary map on the simplicial cochain complex. Prove that $\delta^2 = 0$ or, more precisely, $\delta_{n+1} \circ \delta_n = 0$ for all $n \geq 0$.

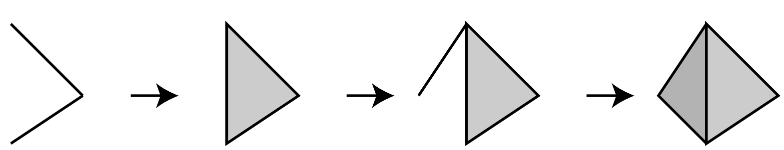
(3) Find a simplicial complex homeomorphic to $\mathbb{R}P^2$.

(4) Find a Δ-complex homeomorphic to the Klein bottle and use it to compute the simplicial homology and cohomology of the Klein bottle.

(5) Let $X_\bullet$ be an abstract simplicial complex and $v_0, \dots, v_n \in X_0$ be vertices of $X_\bullet$ and suppose that there is a $0 \leq k \leq n$ so that $\{v_0, \dots, \widehat{v_k}, \dots, v_n\} \notin X_{n-1}$ but $\{v_0, \dots, \widehat{v_i}, \dots, v_n\} \in X_{n-1}$ for all $i \neq k$. Then we can form a new simplicial complex $Y_\bullet$ with
   (a) $Y_m = X_m$ for $m \notin \{n-1, n\}$,
   (b) $Y_{n-1} = X_{n-1} \cup \{\{v_0, \dots, \widehat{v_k}, \dots, v_n\}\}$, and
   (c) $Y_n = X_n \cup \{\{v_0, \dots, v_k, \dots, v_n\}\}$.

   (In the special case $n = 1$, $Y_\bullet$ is obtained by adding a single new vertex w to X_0 and an edge from some vertex v_0 to w.) We say that $Y_\bullet$ is obtained from $X_\bullet$ by an *n-dimensional elementary expansion* and $X_\bullet$ is obtained from $Y_\bullet$ by an *n-dimensional elementary collapse*. More generally, given abstract simplicial complexes $X_\bullet$ and $Y_\bullet$, we say that $X_\bullet$ is *simple homotopy equivalent* to $Y_\bullet$ if you can get from $X_\bullet$ to $Y_\bullet$ by a sequence of elementary expansions and elementary collapses of any dimension (and re-ordering vertices).

   For example, here is a simple homotopy equivalence between two complexes:



(a) Sketch a proof that if $X_\bullet$ is simple homotopy equivalent to $Y_\bullet$ then their geometric realizations are homotopy equivalent. (The converse is false.)

(b) If $Y_\bullet$ is obtained from $X_\bullet$ is obtained from $X_\bullet$ by an elementary expansion, there is an inclusion map $i_m\colon C_m(X_\bullet) \hookrightarrow C_m(Y_\bullet)$. Construct a quotient map $q_m\colon C_m(Y_\bullet) \to C_m(X_\bullet)$ so that

  (i) the q_m form a chain map,

  (ii) $q_m \circ i_m = \mathbb{I}_{C_m(X_\bullet)}$ and

  (iii) there are maps $h_m\colon C_m(Y_\bullet) \to C_{m+1}(Y_\bullet)$ with

$$i_m \circ q_m - \mathbb{I}_{C_m(Y_\bullet)} = \partial \circ h_m + h_{m-1} \circ \partial.$$

  (Hint: for most simplices σ, you will probably define $h_m(\sigma) = 0$.)

  To save time, it's okay if you only consider the case of an n-dimensional elementary expansion for $n > 1$: the 1-dimensional case is similar, but maybe the notation is a bit different.

(c) Conclude that $H_m(Y_\bullet) \cong H_m(X_\bullet)$ for each m and, consequently, that simple homotopy equivalent simplicial complexes have isomorphic homology groups.

To think about but not turn in:

(1) Given a simplicial map $f\colon X \to Y$, define an induced map $f^*\colon H^n(Y) \to H^n(X)$ and prove your map is well-defined. Prove also that $\mathbb{I}^* = \mathbb{I}$ and if $g\colon Y \to Z$ then $(g \circ f)^* = f^* \circ g^*$.

(2) Give a simplicial complex homeomorphic to $\mathbb{R}P^3$ and compute its simplicial homology and cohomology.

(3) Give a simplicial complex homeomorphic to the n-sphere and compute its simplicial homology and cohomology groups.

Email address: `lipshitz@uoregon.edu`