

# MATH 690 HOMEWORK 4

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- (1) Recall that an *orientation* of an  $n$ -dimensional disk bundle  $D \rightarrow B$  is a choice of generator  $o_b$  of  $H_n(D_b, \partial D_b) \cong \mathbb{Z}$  for each  $b \in B$ , so that for each chart  $(U, \phi: \pi^{-1}(U) \rightarrow U \times D^n)$  for  $B$  and all  $b, b' \in U$ , the compositions

$$H_n(D_b, \partial D_b) \rightarrow H_n(U \times D^n, U \times S^{n-1}) \rightarrow H_n(D^n, S^{n-1}) \quad \text{and}$$

$$H_n(D_{b'}, \partial D_{b'}) \rightarrow H_n(U \times D^n, U \times S^{n-1}) \rightarrow H_n(D^n, S^{n-1})$$

send  $o_b$  and  $o_{b'}$  to the same element.

Show that if  $E$  is an oriented vector bundle with Riemannian metric then the disk bundle  $DE$  inherits an orientation.

- (2) Deduce the Gysin sequence and Thom isomorphism theorem from the Serre spectral sequence.
- (3) Use the Gysin sequence to compute  $H^*(\mathbb{C}P^n)$ , via the fiber bundle  $S^1 \rightarrow S^{2n-1} \rightarrow \mathbb{C}P^n$ .
- (4) Compute the Euler class of the canonical line bundle over  $\mathbb{C}P^n$ , directly from its definition.
- (5) Compute the Euler class of the tangent bundle to  $S^2$ , directly from its definition.
- (6) Given an  $n$ -dimensional disk bundle  $DE$  over a CW complex  $X$  and a section  $s$  of  $SE|_{X^{n-1}}$  over the  $(n-1)$ -skeleton, we associated a cellular cochain  $e_s \in C_{cell}^n(X)$ .
- (a) Show that if  $s'$  is another section of  $SE|_{X^{n-1}}$  then  $e_s - e_{s'}$  is a coboundary.
- (b) Show that for any  $\alpha \in C_{cell}^{n-1}(X)$  there is another section  $s'$  so that  $e_s - e_{s'} = \delta(\alpha)$ .
- (7) Let  $X$  be a finite CW complex. Let  $C(X)$  be the ring of continuous maps  $X \rightarrow \mathbb{R}$ , with pointwise addition and multiplication. Given a vector bundle  $E \rightarrow X$ , the set of sections of  $E$  is a  $C(X)$ -module. Show that the set of sections is a projective  $C(X)$ -module and, conversely, every finitely-generated projective  $C(X)$ -module arises this way. (Hint: use the fact that  $E$  is a subbundle of a trivial bundle.)
- (8)
- (9) Given a closed, oriented, smooth submanifold  $N^n$  of a closed, oriented, smooth manifold  $M^m$ , we have two classes in  $H^*(M)$ :
- (a) The image of the Thom class  $u_{\nu N} \in H^{m-n}(D\nu N, S\nu N) \cong H^{m-n}(M, M \setminus N)$  under the restriction map  $H^{m-n}(M, M \setminus N) \rightarrow H^{m-n}(M)$ .
- (b) The Poincaré dual of the fundamental class of  $N$ ,  $PD[N] \in H^{m-n}(M)$ .
- Prove that these classes are equal.
- (10) Let  $M$  be an oriented, smooth manifold,  $f: M \rightarrow \mathbb{R}$  a Morse function and  $p$  a critical point of  $f$ . Let  $Z$  be the zero-section of  $TM$  and  $\Gamma_{\nabla f}$  the graph of the gradient of  $f$ . Show that the sign of the intersection point  $p \in Z \cap \Gamma_{\nabla f}$  is  $(-1)^{\text{ind}(p)}$ .
- (11) Milnor-Stasheff, Problem 11-C.

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