

MATH 690 HOMEWORK 3

OCTOBER 9, 2017.

ROBERT LIPSHITZ

- (1) With notation as in class, show that G acts on $\mathcal{NEG}$ with quotient $\mathcal{NBG}$.
- (2) Suppose that $\mathcal{C}$ is a category with an initial (or terminal) object. Show that the homology groups of the nerve $\mathcal{NC}$ vanish, and $\pi_1 \mathcal{NC} = 0$, giving another proof that $\mathcal{NC}$ is contractible.
- (3) (Segal, Proposition 2.1.)
 - (a) Suppose that $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor. Show that there is an induced continuous map $\mathcal{NF}: \mathcal{NC} \rightarrow \mathcal{ND}$ of nerves.
 - (b) Suppose that $F_0, F_1: \mathcal{C} \rightarrow \mathcal{D}$ are functors and η is a natural transformation from F_0 to F_1 . Show that $\mathcal{NF}_0$ and $\mathcal{NF}_1$ are homotopic. (Hint: think of η as a map from $\mathcal{C} \times (\bullet \rightarrow \bullet)$ to $\mathcal{D}$.)
 - (c) Give yet another proof that if $\mathcal{C}$ has an initial (or terminal) object then $\mathcal{NC}$ is contractible.
- (4) Prove that the canonical vector bundle γ_n over $\text{Gr}(n, \mathbb{R}^K)$ is, in fact, a vector bundle.
- (5) In class, we gave two definitions of the Bockstein homomorphism:
 - (a) Via the connecting homomorphism associated to the coefficient sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$$

- (b) As the element induced by the generator of $\mathbb{Z}/2 = \text{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/2) = \text{Ext}^1(H_n(K(\mathbb{Z}/2, n)), \mathbb{Z}/2) \subset H^{n+1}(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$ via the Yoneda lemma.

Prove that these two definitions agree.

- (6) *Spin* structures again.
 - (a) Let $f: G \rightarrow H$ be a continuous homomorphism of (reasonable) topological groups. Show that for appropriate choices of BG and BH there is an induced fibration $BG \rightarrow BH$. (Hint: choose any EG and EH and observe that G acts on $EG \times EH$, using f .)
 - (b) Setting $G = \text{Spin}(n)$ and $H = \text{SO}(n)$, what is the fiber of the fibration $B\text{Spin}(n) \rightarrow B\text{SO}(n)$?
 - (c) Show that this fibration is principal (see the section in Hatcher on Postnikov towers).
 - (d) Give another construction of the obstruction w_2 to existence of a *Spin*-structure and another proof that the set of *Spin*-structures on an oriented vector bundle E with $w_2(E) = 0$ is in bijection with $H^1(B; \mathbb{Z}/2)$.

Email address: lipshitz@uoregon.edu