MATH 690 HOMEWORK 1
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(1) With the definition of a vector bundle from class, show that the vector space operations define continuous maps

$$+\colon E \times_B E \to E$$
$$\times\colon \mathbb{R} \times E \to E.$$

(Here, $E \times_B E$ is the fiber product $\{(e_1, e_2) \in E_1 \times E_2 \mid \pi(e_1) = \pi(e_2)\}$.)

(2) Suppose you are given the following data:
- Topological spaces B and F.
- A set E and a map of sets $\pi\colon E \to B$.
- An open cover $\mathcal{U} = \{U_i\}$ of B and for each i a bijection $\phi_i\colon \pi^{-1}(U_i) \to U_i \times F$ so that the following diagram (of sets) commutes:

$$\pi^{-1}(U_i)[rr]^{\phi_i}[dr]_{\pi} U_i \times F[dl]^{\pi_1} U_i$$

Give conditions on the maps ϕ_i so that there is a topology on E making $\pi\colon E \to B$ into a fiber bundle with $\{(U_i, \phi_i)\}$ an atlas.

(3) State and prove the analogue of the previous problem for vector bundles and for principal G-bundles.

(4) Write out the details of the proof that the bundle of frames of a vector bundles is a principal $GL(n)$-bundle.

(5) An *oriented n-dimensional vector bundle* is a vector bundle $\pi\colon E \to B$ together with an orientation of each fiber E_b, so that these orientations are continuous in the following sense. For each $b \in B$ there is a chart $(U \ni b, \phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n)$ so that for all $b' \in U$, $\phi|_{E_{b'}}\colon E_{b'} \to \mathbb{R}^n$ is orientation-preserving. Show that given an oriented n-dimensional vector bundle there is an induced principal $GL_+(\mathbb{R}^n)$-bundle (the "bundle of oriented frames"), and conversely given a principal $GL_+(\mathbb{R}^n)$-bundle there is an induced oriented n-plane bundle.

(6) A *Riemannian metric* on a vector bundle $\pi\colon E \to B$ is an inner product $\langle\cdot,\cdot\rangle_b$ on each fiber E_b of E, which is continuous in the sense that the induced map $E \oplus E = E \times_B E \to \mathbb{R}$ is continuous. Show that given a Riemannian metric on a vector bundle E, there is an induced principal $O(n)$-bundle (the "bundle of orthonormal frames"), and conversely given a principal $O(n)$-bundle there is an induced vector bundle with Riemannian metric.

(7) What operation on principal $O(n)$-bundles corresponds to dualizing a vector bundle? The direct sum of vector bundles?

(8) For nice spaces X (e.g., CW complexes) and abelian groups G, there is a canonical isomorphism $\check{H}^i(X; G) \cong H^i(X; G)$ between the Čech and singular cohomology of X with coefficients in G. A nice, readable proof can be found in Frank Warner's *Foundations of Differential Manifolds and Lie Groups*, Chapter 5. In the rest of this

problem, cohomology either means Čech cohomology or singular cohomology after applying this isomorphism.

(a) Let $\pi\colon E \to B$ be an n-dimensional vector bundle or, equivalently, a principal $GL(n,\mathbb{R})$-bundle, given by a Čech cocycle $\phi \in \check{H}^1(B; GL(n,\mathbb{R}))$. Show that the sign of the determinant $\mathrm{sign}\circ\det\colon\ GL(n,\mathbb{R}) \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ induces a map

$$\check{H}^1(B; GL(n,\mathbb{R})) \to \check{H}^1(B; \mathbb{Z}/2\mathbb{Z})$$

and so ϕ induces an element $w_1(E) \in H^1(B; \mathbb{Z}/2\mathbb{Z})$.

(b) Compute w_1 for the trivial line bundle (1-dimensional vector bundle) over the circle and for the Möbius band.

(c) Prove that (for nice spaces) a line bundle $\pi\colon E \to B$ is trivial if and only if $w_1(E) = 0 \in H^1(B; \mathbb{Z}/2\mathbb{Z})$.

(9) Show that the exact sequence of abelian topological groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 = GL(1,\mathbb{C}) \to 0$$

induces an exact sequence in Čech cohomology

$$\check{H}^1(B; \mathbb{Z}) \to \check{H}^1(B; \mathbb{R}) \to \check{H}^1(B; S^1) \xrightarrow{\delta} \check{H}^2(B; \mathbb{Z}).$$

Given a complex line bundle (principal $GL(1,\mathbb{C})$-bundle) $\pi\colon E \to B$ coming from cocycle data $\phi \in \check{H}^1(B; GL(1,\mathbb{C}))$, let $c_1(E) = \delta(\phi)$. Compute $c_1(E)$ for some complex line bundles over S^2.

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