

MATH 431/531 FALL 2016
HOMEWORK 5
DUE OCTOBER 26, 2016.
 INSTRUCTOR: ROBERT LIPSHITZ

Problems:

- (1) Lee, Exercise 2.42.
- (2) Lee, Exercise 2.51.
- (3) Lee, Problem (not exercise) 2.18.
- (4) Lee, Problem (not exercise) 2.20.
- (5) Lee, Problem (not exercise) 2.22.

Challenge problems (required for Math 531, optional for 431):

- (6) Let $C^0[0, 1]$ denote the set of continuous functions $f: [0, 1] \rightarrow \mathbb{R}$. More generally, for $k \geq 0$ let $C^k[0, 1]$ denote the set of functions $f: [0, 1] \rightarrow \mathbb{R}$ so that f and its first k derivatives (i.e., $f, f', f'', f^{(3)}, \dots, f^{(k)}$) are continuous. Define a metric d_0 on $C^0[0, 1]$ by

$$d_0(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}.$$

(We will prove in a few weeks that this supremum is always finite; you may assume that for now.) Define a metric d_k on $C^k[0, 1]$ by

$$d_k(f, g) = d_0(f, g) + d_0(f', g') + \dots + d_0(f^{(k)}, g^{(k)}).$$

- (a) Prove that $(C^k[0, 1], d_k)$ is a metric space.
- (b) Prove that the topology on $C^k[0, 1]$ induced by d_k is separable.

You may use the following fact, which I will ask you to prove later in the quarter. A *piecewise-linear function* $g: [0, 1] \rightarrow \mathbb{R}$ is a function defined as follows. Fix some $n \in \mathbb{N}$, $0 = a_0 < a_1 < \dots < a_n = 1$, $b_0 \in \mathbb{R}$, and $m_i \in \mathbb{R}$, $i = 0, \dots, n-1$. For $i = 1, \dots, n-1$, define b_i inductively by $b_{i+1} = b_i + m_i(a_{i+1} - a_i)$. Then the function $g = g_{a_0, \dots, a_n; b_0; m_0, \dots, m_{n-1}}$ defined by

$$g(x) = b_i + m_i(x - a_i) \quad \text{if } a_i \leq x \leq a_{i+1}$$

is piecewise-linear. (i.e., the piecewise-linear functions are the ones that can be written in this form.)

Fact (which you may assume). Given $f \in C^0[0, 1]$ and $\epsilon > 0$ then there is $n \in \mathbb{N}$ and $a_0, \dots, a_n, b_0, m_0, \dots, m_{n-1} \in \mathbb{Q}$ (with $0 = a_0 < a_1 < \dots < a_n = 1$) so that $d_0(f, g_{a_0, \dots, a_n; b_0; m_0, \dots, m_{n-1}}) < \epsilon$.

(Hint: integrate repeatedly.)

- (c) Deduce that the topology on $C^k[0, 1]$ induced by d_k is second countable.
- (d) Let $C^\infty[0, 1] = \bigcap_k C^k$ denote the set of functions $f: [0, 1] \rightarrow \mathbb{R}$ so that $f^{(k)}$ is continuous for all $k \geq 0$. Consider the basis

$$\mathcal{B} = \{U \cap C^\infty[0, 1] \mid U \subset C^k[0, 1] \text{ open for some } k \geq 0\}.$$

That is, a set is in $\mathcal{B}$ if it is the intersection of an open set in $C^k[0, 1]$, for some k , with $C^\infty[0, 1]$. Prove that this is a basis for a topology on $C^\infty[0, 1]$. The corresponding topology is called the C^∞ topology.

- (e) Is $C^\infty[0, 1]$ second countable? Separable? First countable? Justify your answers.

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