

MATH 431/531 FALL 2016
HOMEWORK 3
DUE OCTOBER 12, 2016.

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Problems:

- (1) Prove parts (b,d,g) of Proposition 2.8 in Lee.
- (2) Let X be a topological space and A a subset of X . There is a map $i: A \rightarrow X$ given by $i(a) = a$; i is called the *inclusion map*.
 - (a) Prove that if we give A the subspace topology then the inclusion map is continuous.
 - (b) Given topologies $\mathcal{U}, \mathcal{V}$ on A , we say that $\mathcal{V}$ is *finer* than $\mathcal{U}$ if $\mathcal{U} \subset \mathcal{V}$, i.e., if every set which is open with respect to $\mathcal{V}$ is also open with respect to $\mathcal{U}$.

Suppose that $\mathcal{V}$ is a topology on A so that the inclusion map is continuous with respect to $\mathcal{V}$. Prove that $\mathcal{V}$ is finer than the subspace topology. (In other words, the subspace topology is the coarsest topology for which the inclusion map is continuous.)
- (3) Let X be a set. What subsets of X are dense in X with respect to the discrete topology? With respect to the indiscrete topology?
- (4) Let $A = (-5, 0) \cup \{1/n \mid n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$, with the usual (metric) topology. (Here, $\mathbb{Z}_{>0}$ denotes the positive integers. Find:
 - (a) The interior of A .
 - (b) The exterior of A .
 - (c) The closure of A .
 - (d) The boundary of A .
 - (e) The limit points of A .
 - (f) The isolated points of A .No justification needed in this problem; just the answers.
- (5) Consider the unit circle $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$. What is the boundary of the subspace A ? (Justification optional.)
- (6) Find a function $f: \mathbb{R} \rightarrow \mathbb{R}$ (both with the standard topology) which is
 - (a) Discontinuous at every point.
 - (b) Continuous at exactly one point.
 - (c) Continuous exactly at $\mathbb{Z} \subset \mathbb{R}$.(Here, we are using the metric space definition of continuity at a point.)

Challenge problems (required for Math 531, optional for 431):

- (7)
 - (a) Solve Lee Exercise 2.6.
 - (b) Using the previous part, show that if X is a topological space and A is a subspace of X then there is a unique coarsest topology on A so that the inclusion map $i: A \rightarrow X$ is continuous. Do *not* use the explicit description of the subspace topology.

- (8) Consider $(0, 1)^2$ with the lexicographic order, i.e.,

$$(a, b) < (c, d) \equiv \begin{cases} a < c \\ a = c \text{ and } b < d \end{cases} \quad \text{or} \quad .$$

Let $\mathcal{U}$ be the order topology on $(0, 1)^2$ corresponding to this order $<$. Find a metric on $(0, 1)^2$ so that the induced topology is $\mathcal{U}$. (i.e., show that this topology is *metrizable*).

Bonus problems (not required for anyone):

- (9) Consider $[0, 1]^2$ with the lexicographic order. Show that the order topology on $[0, 1]^2$ is *not* metrizable (i.e., is not induced by any metric).
- (10) Let X be a topological space and A a subset of X . The *Cantor derivative* of A is the set of limit points of A .
- (a) Find a nonempty set $A \subset \mathbb{R}$ whose first Cantor derivative is empty.
 - (b) Find a set $A \subset \mathbb{R}$ whose first Cantor derivative is non-empty but whose second Cantor derivative is empty.
 - (c) For any positive integer n , find a set $A \subset \mathbb{R}$ whose $(n - 1)^{\text{st}}$ Cantor derivative is non-empty but whose n^{th} Cantor derivative is empty.
 - (d) The ω^{th} Cantor derivative of A is

$$A^{(\omega)} = A \cap A' \cap A'' \cap A''' \cap \dots$$

(where I am using primes to denote Cantor derivatives). Find a subset $A \subset \mathbb{R}$ so that all finite Cantor derivatives of A are non-empty but $A^{(\omega)}$ is empty.

- (e) Given any countable ordinal α define the α^{th} Cantor derivative of A . (Hint: there are two cases, depending on whether α is a successor ordinal or a limit ordinal.) Find a subset $A \subset \mathbb{R}$ so that if $\beta < \alpha$ then $A^{(\beta)} \neq \emptyset$ but $A^\alpha = \emptyset$.
- (f) A closed subset $A \subset \mathbb{R}$ is *perfect* if $A = A'$. Show that for any closed subset $A \subset \mathbb{R}$ there is a countable ordinal α so that $A^{(\alpha)}$ is perfect.

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