

MATH 431/531 FALL 2016
HOMEWORK 2
DUE OCTOBER 5, 2016.

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Problems:

- (1) Suppose that d is a metric on X , and d_{dis} is the discrete metric on X . Prove that the identity map $Id: (X, d_{dis}) \rightarrow (X, d)$, $Id(x) = x$, is continuous with respect to d_{dis} and d .
- (2) Consider the standard metric d_{std} on $\mathbb{R}$, $d_{std}(x, y) = |x - y|$, and the discrete metric d_{dis} on $\mathbb{R}$. Prove that the identity map $Id: (\mathbb{R}, d_{std}) \rightarrow (\mathbb{R}, d_{dis})$ is not continuous.
- (3) Let (X, d_X) , (Y, d_Y) , and (Z, d_Z) be metric spaces. Give $X \times Y$ the product metric. Suppose that $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ are continuous functions. Prove that $h: Z \rightarrow X \times Y$, $h(z) = (f(z), g(z))$ is a continuous function.
- (4) Suppose that (X, d_X) and (Y, d_Y) are metric spaces, and let $y \in Y$. Prove that the constant function $f: X \rightarrow Y$ given by $f(x) = y$ for all $x \in X$ is continuous.
- (5) In this problem, *continuous* always means with respect to the standard metric on $\mathbb{R}^n$.
 - (a) Prove that the map $a: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $a(x, y) = x + y$ is continuous.
 - (b) Prove that the map $m: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $m(x, y) = xy$ is continuous.
 - (c) Prove that the function $f(x): \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 13x^2 + 5x + 7$ is continuous. Hint: use the previous two parts and the previous two problems.

Challenge problems (required for Math 531, optional for 431):

- (6) Generalizing Problem (5c), prove that if $p(x)$ is any polynomial, viewed as a map $\mathbb{R} \rightarrow \mathbb{R}$, then $p(x)$ is continuous.
- (7) Consider the standard metric d_{std} on $\mathbb{R}^2$ and also the metric d on $\mathbb{R}^2$ given by $d((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\}$.
 - (a) Prove that d is a metric.
 - (b) Describe in words $B_r(p; d) = \{q \in \mathbb{R}^2 \mid d(p, q) < r\}$, the ball of radius r centered at p with respect to d .
 - (c) Let $B_r(p; d_{std}) = \{q \in \mathbb{R}^2 \mid d_{std}(p, q) < r\}$. Prove that $B_r(p; d_{std})$ is open with respect to d .
 - (d) Prove that $B_r(p; d)$ is open with respect to d_{std} .
 - (e) Prove that a set U in $\mathbb{R}^2$ is open with respect to d_{std} if and only if U is open with respect to d .

- (f) Suppose that (X, d_X) is a metric space and $f: \mathbb{R}^2 \rightarrow X$ is a function. Prove that f is continuous with respect to d if and only if f is continuous with respect to d_{std} . (Hint: use the previous part.)

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