

MATH 431/531 FALL 2016
HOMEWORK 1
DUE SEPTEMBER 30, 2016.
INSTRUCTOR: ROBERT LIPSHITZ

Reminder: homework is due at the *beginning* of class, handed to me or in my mailbox on the second floor of Fenton.

Note. This homework assignment is intended partly to make sure you are ready for the course, and is about a third the length, and somewhat easier, than a typical homework assignment.

Math 531 students. This week only it is okay to turn in your homework neatly written, instead of typed in LaTeX.

Problems:

- (1) Let X be a set, and d_{dis} be the *discrete metric* on X , defined by

$$d_{dis}(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

Prove that (X, d_{dis}) is a metric space.

- (2) Consider $\mathbb{R}^2$ with the standard metric $d((x, y), (x', y')) = \sqrt{(x - x')^2 + (y - y')^2}$.
- (a) Prove that $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \text{ and } y > 0\}$ is open in $\mathbb{R}^2$.
- (b) Prove that $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } y \geq 0\}$ is closed in $\mathbb{R}^2$.
- (3) Let X and Y be sets. Show that there is a bijection between the Cartesian product $X \times Y$ and the set of functions

$$\{f: \{1, 2\} \rightarrow X \cup Y \mid f(1) \in X \text{ and } f(2) \in Y\}.$$

Challenge problems (required for Math 531, optional for 431):

- (4) The *2-adic metric* on the set of rationals $\mathbb{Q}$ is defined as follows. Given $n \in \mathbb{Z}$, define $\text{ord}_2(n) = \max\{r \in \mathbb{N} \mid 2^r \text{ divides } n\}$ if $n \neq 0$. For $p/q \in \mathbb{Q}$, define $\text{ord}_2(p/q) = \text{ord}_2(p) - \text{ord}_2(q)$. Then we define

$$d(p/q, r/s) = \begin{cases} 2^{-\text{ord}_2(p/q - r/s)} & \text{if } p/q \neq r/s \\ 0 & \text{if } p/q = r/s. \end{cases}$$

- (a) Compute $d(1, 2)$, $d(1, 3)$, $d(1, 5)$, $d(1, 1/2)$, $d(1/2, 1/3)$.
- (b) Prove that $(\mathbb{Q}, d)$ is a metric space.

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