TOPICS IN PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. These notes were prepared for Math 684 (Topics in Partial Differential Equations) in Fall 2025 at University of Oregon.

Contents

1. Introduction, Motivation, Overview

These are somewhat rough notes. They may not contain complete details for every topic. Let me know if you find anything that seems to be seriously wrong.

1.1. **Brief introduction to the subject of PDE.** A partial differential equation is an equation involving the partial derivatives of an unknown function of several variables. Partial differential equations are ubiquitous in the physical sciences and engineering, appear frequently in other areas of the science, and also play an important role within mathematics more generally. As a mathematical subject, PDE is closely related to the areas of analysis (including functional analysis, harmonic analysis, numerical analysis, spectral theory, and so on), scientific computating, geometry, probability, and others.

PDEs come in many different types. The equations themselves are broadly classified into classes such as elliptic, parabolic, hyperbolic, dispersive, geometric, and may also be categorized as linear, semilinear, quasilinear, fully nonlinear, etc. PDEs are also described by their order, which refers to the highest order derivative appearing in the equation. Particular problems are described as initial-value problems, boundary-value problems, or a mix of these two. A lot of recent work also incorporates probability theory, leading to 'stochastic PDE'.

These notes are not meant to be a comprehensive introduction to PDE. Instead, the emphasis will be on picking some interesting topics, working out some explicit examples, and developing a bit of the theory. We will focus on so-called hyperbolic and dispersive PDE. Elliptic and parabolic equations will be covered in later quarters of this sequence (by a different instructor). Most likely stochastic PDE will not be treated at all in this sequence.

Instead of attempting a general classification, let's just take a look at a few simple representative examples. Many of these will include the Laplacian operator, defined for a function $f : \mathbb{R}^d \to \mathbb{R}$ by

$$\Delta f = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}.$$

It is a remarkable fact that the vast majority of physical phenomena we encounter in our everyday life can be modeled by relatively simple second order PDEs. The derivation of PDE models is an extremely interesting and important topic, although we will not go into this area in any deep way in these notes (indeed, this rich topic really deserves its own course).

Example 1. *The Laplace/Poisson equation takes the form*

$$\begin{cases} -\Delta u = f, \quad x \in \Omega, \\ u = g, \quad x \in \partial\Omega \end{cases}$$

where f, g are given functions and $\Omega \subset \mathbb{R}^d$ is some open set. One can also consider this equation with $\Omega = \mathbb{R}^d$, in which case the boundary data needs to be replaced by some prescribed behavior at infinity. This equation plays a role in classical electrostatics and gravitation. This is a second order, linear, elliptic boundary value problem.

Example 2. *The heat/conduction equation takes the form*

$$\begin{cases} u_t = \Delta u, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \\ u|_{t=0} = f, \quad x \in \mathbb{R}^d. \end{cases}$$

This is an initial-value problem. This is a second order, linear, parabolic problem. This problem and its variants arise in problems of heat conduction and are also related to Brownian motion, finance, image processing, and geometry.

Example 3. *The wave equation (posed on all space) takes the form*

$$\begin{cases} u_{tt} - \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = f, \quad u_t|_{t=0} = g. \end{cases}$$

This is a second order, linear, hyperbolic initial-value problem. This equation and its variants arise in acoustics, electromagnetism, and general relativity.

Example 4. *The Schrödinger equation (with physical constants normalized to 1) takes the form*

$$\begin{cases} iu_t = -\Delta u + Vu, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = f, \end{cases}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ is called the wave function and $V : \mathbb{R}^d \to \mathbb{R}$ is called the external potential. This is a second order, linear, dispersive initial-value problem. This equation is central to quantum mechanics. For example, to model the hydrogen atom one takes $V(x) = |x|^{-1}$. There is a nonlinear counterpart (the nonlinear Schrödinger equation)

$$iu_t = -\Delta u - |u|^2 u,$$

which plays an important role in nonlinear optics.

Example 5. *In many applications, instead of Δu one encounters a term of the form $\nabla \cdot (a(x)\nabla u)$ for some function $a(x)$. For example, in the case of the heat equation, this models heat conduction in an inhomogeneous medium. Similarly, in the wave equation, this models wave propagation in an inhomogeneous medium.*

Most of these examples are posed as *evolution equations*, in which the unknown function depends on variables (t, x) representing time and space, and initial data is prescribed at $t = 0$.

The examples above are all quite simple in structure. This is fine for the purposes of an introduction, but one should be prepared for much more complicated models in real applications.

In all but the most special cases, there is no hope of explicitly solving partial differential equations. In practice, when we study PDE we are firstly interested in questions of *well-posedness* (or *ill-posedness*). Well-posedness means that solutions exist, are unique, and depend in some continuous manner on the data prescribed in the problem. Assuming that one can prove the existence of solutions, it is also of interest to understand the behavior/dynamics of these solutions. In most applications, the only way to really 'solve' a PDE is to use a computer.

2. Notation and preliminaries

Here we introduce notation and collect some analysis results to be used throughout the notes. I tried to keep these notes relatively 'elementary', in the sense that I did not want to assume knowledge of too many advanced topics in functional analysis and harmonic analysis. But of course one cannot really do PDE without analysis, so some advanced topics show up occasionally.

Multiindex notation. Given a multiindex α, we let

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}} = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} u.$$

For k a nonnegative integer, let us denote $\nabla^k u = \{\partial^\alpha u : |\alpha| = 1\}$. When $k = 1$ we view $\nabla^1 u = \nabla u$ (the gradient of u) and when $k = 2$ we view $\nabla^2 u$ as the Hessian matrix of u.

Convolutions, approximate identities, distributions, etc.

Convolution is introduced hastily in Section 7.1 below. In particular,

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy.$$

Convolutions are useful for producing smooth approximations. In particular, $f * g$ is as smooth as the smoother of f and g. Moreover, if g is a fixed smooth L^1 normalized function and $g_n(x) = n^d g(nx)$, then $f * g_n \to f$ in various norms (e.g. L^p-norms). We call this 'approximation to the identity'.

It is often useful to interpret various statements 'in the sense of distributions'. As an example, instead of asking that u solve the PDE $-\Delta u = f$ in the 'classical sense', we might instead ask that

$$-\int u\Delta\varphi = \int f\varphi$$

for all φ chosen from a very nice but still sufficiently 'rich' subclass, like C_c^∞ or Schwartz space. In the event that u itself is smooth, the identity above guarantees that u satisfies the PDE in the classical sense (by using integration by parts an approximation to the identity argument). But the formulation above puts the burden of being smooth on the 'test function' φ, and makes sense *a priori* for a very large class of possible u (e.g. locally integrable functions).

In general, we say a distribution is a continuous linear functional on test functions. Continuity should be defined relative to the topology you put on test functions. Locally integrable functions can be identified by distributions by identifying the function f with the distribution u_f defined via

$$u_f(\varphi) = \int f\varphi\,dx.$$

Once you make this identification, you can do a lot of things that you couldn't do before, like differentiate a non-differentiable function. This particular idea is based on the integration by parts formula

$$\int f'\varphi = -\int f\varphi'.$$

Thus if we have a function f for which the right-hand side above makes sense for any test function φ, we can declare that f' exists as a distribution, and the formula above tells us how to compute it.

One special example is the delta distribution, defined by $\delta_0(\varphi) = \varphi(0)$. If we define f to be the characteristic function of $(0,\infty)$, then we find that f *is* differentiable (in the sense of distributions), and its derivative is δ_0. *Exercise*: check this!

Some results from calculus and analysis.

In this section, we will collect some results from calculus and analysis that we will need at various points throughout the notes.

Theorem 2.1 (Implicit Function Theorem, [6])**.** *Let X, Y, Z be Banach spaces and $\mathcal{O} \subset X \times Y$ an open set. Suppose $F : \mathcal{O} \to Z$ is a C^1 mapping and that $(x_0, y_0) \in \mathcal{O}$ is such that $F(x_0, y_0) = 0$ and $D_yF(x_0, y_0)$ has a bounded inverse in $\mathcal{L}(Z, Y)$.*

Then there exists a neighborhood $U \times V$ of (x_0, y_0) contained in $\mathcal{O}$ and C^1 mapping $g : U \to V$ such that

$$y_0 = g(x_0) \quad \textit{and} \quad F(x, g(x)) = 0 \quad \textit{for all} \quad x \in U.$$

Moreover, if $F(x,y)=0$ for $(x,y)\in U\times V$, then $y=g(x)$.

Theorem 2.2 (Inverse Function Theorem, [2]). *Let $x_0\in U\subset\mathbb{R}^n$, with U open, and let $z_0=f(x_0)$. If $f\in C^1(U;\mathbb{R}^n)$ and $\nabla f(x_0)$ is nondegenerate, then there exists an open set $x_0\in V\subset U$ and an open set $z_0\in W\subset\mathbb{R}^n$ such that $f:V\to W$ is bijective with $f^{-1}\in C^1$.*

Theorem 2.3 (Contraction Mapping Principle). *Let (X,d) be a complete metric space. Suppose $\Phi:X\to X$ and that Φ satisfies*

$$d(\Phi(u),\Phi(v))\le\tfrac{1}{2}d(u,v)\quad\text{for all}\quad u,v\in X.$$

Then there exists unique $u\in X$ such that $u=\Phi(u)$.

Proof. Exercise. □

I found the following lemma stated in this form on Wikipedia. It is used once below in a 'compactness argument'.

Lemma 2.1 (Aubin–Lions–Simon Lemma). *Suppose $X_0\subset X\subset X_1$ are three Banach spaces with X_0 compactly embedded in X and X continuously embedded in X_1. For $1\le p,q\le\infty$ define*

$$W=\{u\in L^p([0,T];X_0):\partial_t u\in L^q([0,T];X_1)\}.$$

If $p<\infty$ then the embedding of W into $L^p([0,T];X)$ is compact. If $p=\infty$ and $q>1$, then the embedding of W into $C([0,T];X)$ is compact.

3. Some ODE Theory

Reference: Sideris.

Some approaches to solving PDE entail finding some way to reduce the PDE to an ODE system, at which point one can either solve the ODEs explicitly or appeal to some ODE existence theory. This section will collect a few basic results from ODE that may be of use in later sections. A good textbook covering a lot of the theory (including general existence results) is Sideris [6].

Everybody should know how to define an integrating factor for a first order, linear ODE. First consider a homogeneous equation of the form

$$y'(t)+a(t)y(t)=0.$$

If we define $\Phi(t)=\int_0^t a(s)\,ds$, then the equation above is equivalent to

$$\tfrac{d}{dt}\big[e^{\Phi(t)}y(t)\big]=0.$$

This can be integrated:

$$e^{\Phi(t)}y(t)\equiv C,\quad\text{so that}\quad y(t)=e^{-\Phi(t)}C.$$

The value C is determined by $C=y(0)$.

Now consider the more general inhomogeneous case

$$y'(t)+a(t)y(t)=f(t).$$

Use the same integrating factor $e^{\Phi(t)}$ as above. We derive

$$\tfrac{d}{dt}\big[e^{\Phi(t)}y(t)\big]=e^{\Phi(t)}f(t).$$

This can still be integrated! Incorporating the initial condition leads to

$$y(t) = e^{-\Phi(t)}y(0) + \int_0^t e^{-\Phi(t)}e^{\Phi(s)}f(s)\,ds.$$

Everyone should also be able to solve a separable equation. Let us run through the basic idea here. If we can write $\frac{dy}{dx} = f(x)g(y)$, then we can formally separate

$$\tfrac{dy}{g(y)} = f(x)\,dx.$$

Integrating both sides yields at least an implicit formula for y in terms of x.

4. Cauchy–Kovalevskaya Theorem

Reference: Evans Section 4.6

I wanted in these notes to cover at least one very general existence theorem for PDE. We consider the following first-order PDE system for an unknown $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$:

$$\begin{cases} u_t = B(u,y)\cdot\nabla_y u + c(u,y) & |(t,y)| < r \\ u(0,y) = 0 & |y| < r. \end{cases} \tag{4.1}$$

Here $r > 0$, $u = u(t,y) \in \mathbb{R}^m$ with $(t,y) \in \mathbb{R}\times\mathbb{R}^n$, $c : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, and we use the notation

$$B(u,y)\cdot\nabla_y u = \sum_{j=1}^n B_j(u,y)u_{y_j},$$

where subscripts denote partial differentiation and $B_j : \mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}^{m\times m}$ (so that $B\cdot\nabla_y u \in \mathbb{R}^m$).

Remark 4.1. *One can obtain this system from a more general quasilinear equation. (Exercise, or check [2].)*

The equations for the individual components of u are written as

$$u_t^k = \sum_{j=1}^n \sum_{\ell=1}^m B_j^{k\ell}(u,y)u_{y_j}^\ell + c^k(u,y). \tag{4.2}$$

Here we use subscripts to denote partial derivatives. For example,

$$u_{ty_1} = \tfrac{\partial^2 u}{\partial t \partial y_1} \quad\text{and}\quad B_{j;y_i}^{k\ell} = \tfrac{\partial B_j^{k\ell}}{\partial y_i}.$$

Our interest is a general local existence result in the case of analytic coefficients and data. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *analytic* (near a point $x_0 \in \mathbb{R}^n$) if there exist $r > 0$ and $\{f_\alpha\} \subset \mathbb{R}$ such that

$$f(x) = \sum_\alpha f_\alpha\,(x - x_0)^\alpha$$

for $|x - x_0| < r$. Here the sum is taken over all multiindices α. If f is analytic, then necessarily

$$f_\alpha = \frac{\partial^\alpha f(x_0)}{\alpha!}.$$

Our goal will be to prove the following:

Theorem 4.1 (Cauchy–Kovalevskaya)**.** *Suppose that B and c are analytic. Then there exist $r > 0$ and a real analytic function u on $|(t,y)| < r$ solving* (4.1).

Proof. Writing $x = (t, y)$, we wish to construct a solution of the form

$$u(x) = \sum_\alpha u_\alpha x^\alpha, \quad u_\alpha = \frac{\partial^\alpha u(0)}{\alpha!}.$$

We will determine what the coefficients u_α must be in terms of B and c and prove that the corresponding power series converges on some ball $|x| < r$.

By assumption, there exists some $s > 0$ so that the functions B and c admit a power series expansion in (z, y) for $|z| + |y| < s$:

$$B_j(z, y) = \sum_{\gamma,\delta} B_{j,\gamma,\delta} z^\gamma y^\delta, \quad c(z, y) = \sum_{\gamma,\delta} c_{\gamma,\delta} z^\gamma y^\delta,$$

with

$$B_{j,\gamma,\delta} = \frac{\partial_z^\gamma \partial_y^\delta B_j(0,0)}{(\gamma+\delta)!}, \quad c_{\gamma,\delta} = \frac{\partial_z^\gamma \partial_y^\delta c(0,0)}{(\gamma+\delta)!}.$$

By the boundary condition $u \equiv 0$ on $\{t = 0\}$, we have

$$u_\alpha = \tfrac{\partial^\alpha u(0)}{\alpha!} = 0 \quad \text{whenever} \quad \alpha = (0, \beta). \tag{4.3}$$

Now let us start from (4.2) and differentiate with respect to y_i for some $i \in \{1, \dots, n\}$. This leads to

$$u^k_{ty_i} = \sum_{j,\ell} \left[B_j^{k\ell} u^\ell_{y_j y_i} + B^{k\ell}_{j;y_i} u^\ell_{y_j} + \sum_{p=1}^m B^{k\ell}_{j;z_p} u^p_{y_i} u^\ell_{y_j} \right] + c^k_{y_i} + \sum_{p=1}^m c^k_{z_p} u^p_{y_i}.$$

Evaluating at $x = 0$ (and recalling (4.3)) leads to

$$u^k_{ty_i}(0) = c^k_{y_i}(0,0), \quad i = 1, \dots, n,$$

which can be written

$$\partial_x^\alpha u(0) = \partial_y^\beta c(0,0) \quad \text{whenever} \quad \alpha = (1, \beta), \quad |\beta| = 1.$$

In fact, we claim that for any multiindex of the form $\alpha = (1, \beta)$ we have

$$\partial_x^\alpha u(0) = \partial_y^\beta c(0,0), \quad \alpha = (1, \beta).$$

Indeed, we again differentiate the equation (and will evaluate at zero):

$$\partial_x^\alpha u^k = \sum_{j,\ell} \partial_y^\beta [B_j^{k\ell} u^\ell_{y_j}] + \partial_y^\beta [c^k].$$

At this point, one would need to apply the Leibniz rule to expand further. However, we can already note the terms involving B_j always result in a multiindex γ with $\gamma_1 = 0$ in the u term, so that by (4.3) they vanish at $x = 0$. Similarly, after applying the chain rule to the $\partial_y^\beta [c^k(u, y)]$ term, any term involving the derivative of u will vanish by (4.3). Thus the claim follows.

Next let us consider multiindices of the form $\alpha = (2, \beta)$. We use the equation to compute

$$\begin{aligned}
\partial_x^\alpha u^k &= \partial_y^\beta \partial_t [u_t^k] \\
&= \partial_y^\beta \partial_t \left[\sum_{j,\ell} B_j^{k\ell} u^\ell_{y_j} + c^k \right] \\
&= \partial_y^\beta \left[\sum_{j,\ell} \left(B_j^{k\ell} u^\ell_{ty_j} + \sum_{p=1}^m B^{k\ell}_{j;z_p} u^p_t u^\ell_{y_j} \right) + \sum_{p=1}^m c^k_{z_p} u^p_t \right].
\end{aligned}$$

Evaluating at $x=0$ (and recalling (4.3)) yields

$$\partial_x^\alpha u^k(0) = \partial_y^\beta \Big[\sum_{j,\ell} B_j^{k\ell} u_{ty_j}^\ell + \sum_{p=1}^m c_{z_p}^k u_t^p \Big] \Big|_{(x,u)=(0,0)}.$$

If we now work out the right-hand side, we find that it is a polynomial in the derivatives of B_j, the derivatives of c, and derivatives $\partial_x^\gamma u$ with $\gamma_1 \leq 1$. Moreover, the coefficients of this polynomial are all nonnegative. As we have already determined $\partial_x^\gamma u$ for such multiindices, it follows that we can determine $\partial_x^\alpha u^k(0)$ for any multiindex of the form $\alpha = (2,\beta)$.

In general, we can derive that

$$\partial_x^\alpha u^k(0) = p_\alpha^k(\dots, \partial_z^\gamma \partial_y^\delta B_j, \dots, \partial_z^\gamma \partial_y^\delta c, \dots, \partial_x^\beta u, \dots)|_{(x,u)=(0,0)},$$

where p_α^k is a polynomial with nonnegative coefficients and $\beta_1 \leq \alpha_1 - 1$ for any multiindex appearing on the right. Thus

$$u_\alpha^k = q_\alpha^k(\dots, B_{j,\gamma,\delta}, \dots, c_{\gamma,\delta}, \dots, u_\beta, \dots)$$

(where $\beta_1 \leq \alpha_1 - 1$), which allows us to recursively determine the coefficients in the power series of u.

At this point we have shown that if an analytic solution exists, then we can determine the coefficients in its power series. We will next show that the power series defined by these coefficients actually converges in some ball. Then, since the Taylor series of u_t and $B \cdot \nabla_y u + c$ agree at $x=0$, they agree everywhere in this ball (i.e u solves the PDE).

To prove convergence of the power series, we utilize the *method of majorants.* Our presentation will be brief.

Given $r>0$ and $C \geq 1$, we let

$$\psi_r(z,y) = \frac{r}{r - (y_1 + \cdots + y_n) - (z_1 + \cdots + z_m)}$$

and define

$$B_j^*(z,y) = C\psi_r(z,y)\mathbf{1} \quad \text{and} \quad c^*(z,y) = C\psi_r(z,y)\,\mathbf{1},$$

where $\mathbf{1}$ denotes the matrix/vector of all ones. If C is sufficiently large, then B_j^* and c^* will majorize B_j and c in a sufficiently small ball (in the sense that the power series coefficients of the former are greater than the magnitude of the coefficients of the latter, component-wise). This relies on the fact that

$$\frac{s}{s - (x_1 + \cdots + x_n)} = \sum_\alpha \frac{|\alpha|!}{s^{|\alpha|}\alpha!} x^\alpha, \quad |x| < \frac{s}{\sqrt{n}}.$$

Now consider the problem

$$u_t^* = B(u^*, y) \cdot \nabla_y u^* + c^*(u^*, y) = 0, \quad u^*|_{t=0} = 0.$$

This problem has an explicit solution! It is $u^* = v^* \mathbf{1}$, where

$$v^*(x) = \tfrac{1}{m[n+1]} \big\{ r - (y_1 + \cdots + y_n) - [(r - (y_1 + \cdots + y_n))^2 - 2m(n+1)Crt]^{\frac12} \big\},$$

which is analytic on a sufficiently small ball.

On the other hand, using the same arguments as above, we can also work out the coefficients of the power series of u^* recursively, which will lead to the consideration of exactly the same polynomials q_α^k as above.

Since these polynomials have all nonnegative coefficients, one can use induction to show that $|u^k_\alpha| \le u^{k*}_\alpha$ for all coefficients. This relies on the fact that for a polynomial with nonnegative coefficients and $|a|, |b|, |c| \le A, B, C$, we have

$$|q(a, b, c, \dots)| \le q(|a|, |b|, |c|, \dots) \le q(A, B, C, \dots).$$

In particular, we can deduce that u^* majorizes u, and hence u converges in the same ball as u^*. The result follows. □

Theorem 4.1 assumed zero data $t = 0$. This can easily be replaced with an arbitrary analytic function:

Corollary 4.1. *Consider the following system for $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$:*

$$\begin{cases} u_t = B(u, y) \cdot \nabla_y u + c(u, y) & |(t, y)| < r \\ u(0, y) = g(y) & |y| < r, \end{cases} \tag{4.4}$$

where B, c, g are analytic in some ball around 0. Then there exists $r > 0$ and a real analytic solution u on $|(t, y)| < r$.

Proof. Fix analytic g. Define

$$\tilde{B}(v, y) = B(v + g, y) \quad \text{and} \quad \tilde{c}(v, y) = B(v + g, y) \cdot \nabla_y g + c(v + g, y).$$

Theorem 4.1 yields an analytic solution v to

$$\partial_t v = \tilde{B}(v, y) \cdot \nabla_y v + \tilde{c}(v, y), \quad v|_{t=0} = 0.$$

Then $u := v + g$ solves (4.4). □

We remark that Theorem 4.1 may be used to obtain existence for more general quasilinear PDE (with analytic coefficients), since these equations can be brought into the form of a first order system (like the system we considered above). Moreover one can also obtain existence for fully nonlinear (analytic) PDE. The interested reader should open [2] for more information.

5. First-order scalar equations, method of characteristics

Reference: Whitham Ch. 2, Evans Ch. 3

Let us begin by introducing a simple first-order, one-dimensional model for wave propagation involving a continuous distribution of some material. Specific examples will be discussed below.

We define a density $\rho(t, x)$ per unit length and a flux $q(t, x)$ per unit time. (The flow velocity is then $v = q/\rho$.) Assuming the material is conserved, we impose that the rate of change of total amount of material in (x_1, x_2) is balanced by the net flow across x_1 and x_2:

$$\tfrac{d}{dt} \int_{x_1}^{x_2} \rho(t, x)\, dx + q(t, x_2) - q(t, x_1) = 0. \tag{5.1}$$

In the limit $|x_2 - x_1| \to 0$ (assuming the functions are smooth), this leads to the *conservation law*

$$\rho_t + q_x = 0.$$

In simple settings, we may assume a functional relationship of the form $q = Q(p)$, so that the equation above reduces to the nonlinear wave equation

$$\rho_t + c(\rho)\rho_x = 0, \quad \text{with} \quad c = Q'.$$

The simplest possible case occurs when $c(\rho)$ is constant:

$$\rho_t + c_0\rho_x = 0, \quad c_0 = \text{constant}.$$

As one can check via direct computation, the solution in this case is

$$\rho(t,x) = f(x - c_0 t),$$

where $f(x) = \rho(0,x)$ (the initial condition for ρ).

The nonlinear counterpart

$$\rho_t + c(\rho)\rho_x = 0$$

is much more interesting and its study leads to many of the essential ideas in nonlinear hyperbolic waves. As before, let us denote the initial condition by $f(x) = \rho(0,x)$.

Let us look for a solution using the *method of characteristics*. This entails looking for a curve $x = x(t)$ along which the PDE reduces to an ODE that we can solve. Given some curve $x = x(t)$ and assuming $\rho = \rho(t,x)$ is a solution, let us define

$$z(t) = \rho(t, x(t)).$$

Then (using $\dot{}$ to denote time derivative)

$$\dot z(t) = \rho_t(t, x(t)) + \dot x(t)\rho_x(t, x(t)).$$

It follows from the PDE that if

$$\dot x(t) = c(\rho(t, x(t)) = c(z(t)),$$

then $\dot z \equiv 0$. Then in fact

$$\dot x \equiv c(\rho(0, x(0))) = c(f(x(0))) \quad \text{and} \quad \rho(t, x(t)) \equiv f(x(0)).$$

In particular, for any $\xi = x(0) \in \mathbb{R}$, we can define the characteristic curve $x(t) = \xi + tc(f(\xi))$ and obtain

$$\rho(t, \xi + c(f(\xi))t) = f(\xi).$$

Now, given $(t,x) \in \mathbb{R}\times\mathbb{R}$, we look for $\xi = \xi(t,x) \in \mathbb{R}$ such that $x = \xi + c(f(\xi))t$. If this is possible, we obtain

$$\rho(t,x) = f(\xi), \quad \text{where} \quad \xi = \xi(t,x) \quad \text{satisfies} \quad x = \xi + c(f(\xi))t.$$

Does this solve our problem? The initial condition is certainly satisfied, since $\xi(0,x) = x$. As for the PDE, we first use the chain rule to obtain

$$\rho_t + c(\rho)\rho_x = f'(\xi)[\xi_t + c(f(\xi))\xi_x].$$

Using implicit differentiation in the formula $x = \xi + c(f(\xi))t$, we obtain

$$\xi_t = -\frac{c(f(\xi))}{1 + tc'(f(\xi))f'(\xi)} \quad \text{and} \quad \xi_x = \frac{1}{1 + tc'(f(\xi))f'(\xi)},$$

so that $\xi_t + c(f(\xi))\xi_x = 0$. Thus the PDE is satisfied as well. The method looks promising.

However, there can be serious issues with this construction. Let us demonstrate this with some examples.

Example 6. *Suppose* $c(\rho) = \rho$ *and*

$$f(x) = \begin{cases} 1 & x \leq 0, \\ 1 - x & 0 \leq x \leq 1, \\ 0 & x \geq 1. \end{cases}$$

The solution obtained above is written

$$\rho(t, x) = f(\xi), \quad \textit{where} \quad \xi = \xi(t, x) \quad \textit{satisfies} \quad x = \xi + f(\xi)t.$$

Thus the characteristics are:

$$\begin{aligned} &t = x - \xi \quad \textit{if} \quad \xi \leq 0, \\ &t = \tfrac{x-\xi}{1-\xi} \quad \textit{if} \quad 0 \leq \xi < 1, \\ &x = \xi \quad \textit{if} \quad \xi \geq 1. \end{aligned}$$

In particular, the characteristics intersect at $t = 1$ *(and* $x = 1$*), and our solution method breaks down. For times* $0 \leq t \leq 1$*, however, we can obtain*

$$\rho(t, x) = \begin{cases} 1 & x \leq t, \\ \frac{1-x}{1-t} & t \leq x \leq 1, \\ 0 & x \geq 1. \end{cases}$$

So what is going on at times $t \geq 1$*? Our method seems to produce some kind of multi-valued solution, which we must reject. Can we continue the solution past* $t = 1$ *in a meaningful (i.e. physical) way?*

Example 7. *Suppose* $c(\rho) = \rho$ *and*

$$f(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0. \end{cases}$$

The characteristics satisfy $x = \xi + f(\xi)t$*, so that*

$$\begin{aligned} &t = x - \xi \quad \textit{if} \quad \xi < 0, \\ &x = \xi \quad \textit{if} \quad \xi \geq 0. \end{aligned}$$

Characteristics begin to cross immediately — so we do not seem to obtain a solution for any amount of time! (Note that even if we take f *to be a slightly smoothed version of the function above, this issue will persist.)*

Example 8. *Suppose* $c(\rho) = \rho$ *and*

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

The solution we obtained above is written

$$\rho(t, x) = f(\xi), \quad \textit{where} \quad \xi = \xi(t, x) \quad \textit{satisfies} \quad x = \xi + f(\xi)t.$$

Thus characteristics are

$$\begin{aligned} &x = \xi \quad \textit{if} \quad \xi \leq 0, \\ &t = x - \xi \quad \textit{if} \quad \xi > 0. \end{aligned}$$

We therefore obtain

$$\rho(t,x) = \begin{cases} 0 & x < 0, \\ 1 & 0 \le t \le x. \end{cases}$$

However, we obtain no candidate for solution in the region $\{0 \le x \le t\}$ *via this method, as no characteristics enter this region. In fact, there is more than one way to define a solution in the remaining region. Is there some criterion to pick out the 'right' solution?*

When characteristics cross, it seems that our equation has somehow failed as a physical model. One option is to return to the physics and develop an improved model. In the present case, we can still obtain a reasonable solution by allowing for discontinuities in the equation. It is important, however, that we do not just find some *ad hoc* mathematical solution. Instead, we should find a solution that is compatible with the underlying physics. In particular, the relevant physics is related to the formation of shock waves, i.e. thin regions in which the flow variables change rapidly. In the simple model we are considering, the shock region corresponds to a discontinuity. (Below we will also resolve the issue presented in Example 8 through other considerations.)

To proceed, let us go back to the original conservation law

$$\tfrac{d}{dt}\int_{x_1}^{x_2} \rho(t,x)\,dx + q(t,x_2) - q(t,x_1) = 0,$$

this time supposing that there is some discontinuity at some $x = s(t) \in (x_1, x_2)$ (where the characteristics will cross). Splitting the integral into $\int_{x_1}^{s(t)}$ and $\int_{s(t)}^{x_2}$, this leads to

$$[\rho^-(t,s(t)) - \rho^+(t,s(t))]\dot{s}(t) + \int_{x_1}^{x_2} \rho_t(t,x)\,dx + q(t,x_2) - q(t,x_1) = 0,$$

where $\rho^\pm$ denotes limits from the right/left. In the limit as $x_1 \to s(t)^-$ and $x_2 \to s(t)^+$, this leads to

$$q^+(t,s(t)) - q^-(t,s(t)) = [\rho^+(t,s(t))] - \rho^-(t,s(t))]\dot{s}(t),$$

which we abbreviate as

$$[q] = [\rho]\dot{s}. \tag{5.2}$$

This can be read as a condition for specifying the discontinuity curve $x = s(t)$. (It may be called the *Rankine–Hugoniot* condition.)

With this in mind, let us return to Example 6:

Example 9 (Example 6 - continued)**.** *Recall that* $c(\rho) = \rho$*, which corresponds to* $q = Q(\rho) = \frac{1}{2}\rho^2$*. The characteristics begin to cross at* $(t,x) = (1,1)$*. The value of the solution is* 1 *on the left and* 0 *on the right. Thus*

$$[\rho] = -1 \quad \textit{and} \quad [q] = -\tfrac{1}{2},$$

so that (5.2) *dictates* $\dot{s} \equiv \frac{1}{2}$*. Since* $s(1) = 1$*, we therefore define*

$$s(t) = 1 + \tfrac{1}{2}(t-1) = \tfrac{1}{2}(1+t).$$

For $t > 1$*, we obtain the shock solution*

$$\rho(t,x) = \begin{cases} 1 & x < \frac{1}{2}(1+t) \\ 0 & x > \frac{1}{2}(1+t). \end{cases}$$

Example 7 is treated similarly.

Example 10 (Example 2 - continued)**.** *We need to define the shock for all $t > 0$, as the characteristics cross at $(0, 0)$. The value of the jump is again $\frac{1}{2}$, so that we should take $s(t) = \frac{1}{2}t$. The shock solution is*

$$\rho(t,x) = \begin{cases} 1 & x < \frac{1}{2}t \\ 0 & x > \frac{1}{2}t. \end{cases}$$

These shock solutions, although discontinuous, define reasonable PDE solutions that are compatible with the original physical formulation of the problem. (We will discuss some alternate approaches to defining the shock condition below.)

There is another condition that should be included in order to properly call a curve of discontinuities a *shock*. This is known as the *entropy conidition*. It amounts to imposing the condition that if one travels backward in time along any characteristic, one will not encounter any discontinuities. In the case that Q is uniformly convex, this ultimately amounts to imposing that the value to the left of the shock exceeds the value to the right. For more details see [2, Section 3.4.1].

What about the situation in Example 8? Recall in this case that there was a region of the (x, t) plane that was not reached by any characteristics.

Example 11 (Example 8 - continued)**.** *Return to Example 8. We should not define a shock because it would violate the entropy condition. Instead, we present the following approach.*

Take $0 < \varepsilon \ll 1$ and consider the modified initial data $f_\varepsilon(x)$ defined by

$$f_\varepsilon(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{\varepsilon} & x \in [0, \varepsilon], \\ 1 & x \geq \varepsilon. \end{cases}$$

Then $f_\varepsilon \to f$ as $\varepsilon \to 0$. We will construct solutions ρ_ε with data f_ε and see if we can obtain a limit as $\varepsilon \to 0$. As above, we obtain

$$\rho_\varepsilon(t,x) = f_\varepsilon(\xi), \quad \textit{where} \quad \xi = \xi(t,x) \quad \textit{satisfies} \quad x = \xi + f_\varepsilon(\xi)t.$$

The characteristics are thus

$$\begin{aligned} &x = \xi \quad \textit{if} \quad \xi \leq 0, \\ &t = \tfrac{\varepsilon}{\xi}x - \varepsilon \quad \textit{if} \quad \xi \in (0, \varepsilon], \\ &t = x - \xi \quad \textit{if} \quad \xi > \varepsilon. \end{aligned}$$

This leads to the continuous solution

$$\rho_\varepsilon(t,x) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} - \varepsilon\frac{x}{t(t+\varepsilon)} & 0 \leq x \leq t + \varepsilon \\ 1 & x \geq t + \varepsilon. \end{cases}$$

We can now send $\varepsilon \to 0$ to obtain

$$\rho(t,x) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 \leq x \leq t \\ 1 & x \geq t. \end{cases}$$

This defines a continuous solution to the original problem! We call this solution a rarefaction wave.

To help give more intuition for the ideas above, let us discuss one simple, interesting application to traffic flow.

Example 12 (Traffic flow)**.** *We model traffic flow by treating a stream of traffic as a continuum with density $\rho(t,x)$ (number of cars per unit length) and flux $q(t,x)$ (number of cars crossing position x per unit time). On a stretch of highway without exits/onramps, the total number of cars is conserved, so that we may impose* (5.1)*.*

We may also assume as a first approximation a relationship of the form $q = Q(\rho)$. Taking $Q(\rho) = \frac{1}{2}\rho^2$ yields the equation studied above, but this doesn't quite make sense for traffic flow (as it corresponds to the assumption that flow velocity is proportional to density). We might instead suppose

$$Q(\rho) = \rho(1-\rho),$$

corresponding to flow velocity $v = 1-\rho$. (Here we implicitly work in units so that there is a maximum possible density and velocity equal to 1.) This is known as the Lighthill–Whitham–Richards model, and it leads to the equation

$$\rho_t + c(\rho)\rho_x = 0, \quad c(\rho) = 1 - 2\rho.$$

Writing $f(x) = \rho(0,x)$, the solution we obtained above is written

$$\rho(t,x) = f(\xi), \quad \textit{where} \quad \xi = \xi(t,x) \quad \textit{satisfies} \quad x = \xi + [1-2f(\xi)]t.$$

Now consider some special cases of $f(x)$ as above. Let us work with

$$f(x) = \begin{cases} \alpha & x < 0 \\ \beta & x > 0, \end{cases} \quad (0 \leq \alpha, \beta < 1).$$

Let us consider $\alpha < \beta$, corresponding to 'denser traffic ahead'. The characteristics are

$$\begin{aligned} t &= \tfrac{x-\xi}{1-2\alpha} \quad \textit{if} \quad \xi \leq 0, \\ t &= \tfrac{x-\xi}{1-2\beta} \quad \textit{if} \quad \xi > 0. \end{aligned}$$

Characteristics cross immediately at $x = 0$. This leads to a shock, which we denote by $x = s(t)$ as above. To the left of the shock, the solution has value α, while to the right of the shock, the solution has value β. Imposing the jump condition (5.2) *(with $q = \rho(1-\rho)$) leads to*

$$\dot{s} = 1 - (\alpha + \beta).$$

Thus we obtain the solution

$$\rho(t,x) = \begin{cases} \alpha & x < [1-(\alpha+\beta)]t \\ \beta & x > [1-(\alpha+\beta)]t. \end{cases}$$

In particular, if $\alpha + \beta > 1$ then the shock moves 'backwards'—a traffic jam!

Next suppose $\alpha > \beta$, corresponding to 'lighter traffic ahead'. The characteristics are the same, but this time there is a region in the (x,t) plane that does not intersect any characteristics. Thus we construct a rarefaction wave as before. Using the same method as above, we obtain

$$\rho(t,x) = \begin{cases} \alpha & x < (1-2\alpha)t \\ \frac{1}{2} - \frac{x}{2t} & (1-2\alpha)t \leq x \leq (1-2\beta)t \\ \beta & x > (1-2\beta)t. \end{cases}$$

Let us give a few alternate points of view concerning the derivation of the shock condition.

Example 13 (Weak solutions)**.** *The presence of discontinuities in a function that is meant to solve a PDE might make you uncomfortable. A remedy for this is to utilize the theory of distributions, or the closely-related notion of weak solutions. For example, we can introduce the more general notion of an* integral solution *to* $\rho_t + \partial_x Q(\rho) = 0$ *with* $\rho(0,x) = f(x)$ *to be a bounded function such that*

$$\int_0^\infty \int_{\mathbb{R}} \rho\varphi_t + Q(\rho)\varphi_x \, dx\, dt + \int_{\mathbb{R}} f\, \varphi(0,\cdot)\, dx = 0$$

for every $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$*. Assuming that there is some curve of discontinuity* $x = s(t)$ *as above, this formulation may be used to derive the same shock condition as before. Moreover, the (discontinuous) shock solutions that we constructed above satisfy the PDE in this weaker formulation. See [2].*

Example 14. *Let us return to the traffic flow example. The assumption* $q = Q(\rho)$ *may be an oversimplification. A slightly better assumption is*

$$q = Q(\rho) - \nu\rho_x, \quad \nu > 0.$$

This reflects the fact that drivers will reduce their speed to account for an increasing density ahead. This choice results in a quite different PDE, namely

$$\rho_t + c(\rho)\rho_x = \nu\rho_{xx}, \quad c(\rho) = Q'(\rho).$$

If ρ_x *is small, we may expect that the two models behave similarly; however, in the case of wave breaking this term becomes huge and the correction term is crucial. By looking for a solution with step-like data of the form* $\rho(t,x) = g(x - at)$ *(for* a *to be determined), one can derive the same jump condition we obtained above (namely,* a *corresponds to* $\dot{s}$*). See [8].*

We turn to a more general quasi-linear equations of the form

$$\rho_t + c\rho_x = b,$$

where c, b are functions of ρ, t, and x. We impose an initial condition $\rho(0,x) = f(x)$ as before.

Once again, let us try the method of characteristics. We will try to solve the PDE along some curve $x = x(t)$. Setting $z(t) = \rho(t, x(t))$, we obtain (as before)

$$\dot{z}(t) = \rho_t(t, x(t)) + \dot{x}(t)\rho_x(t, x(t)).$$

Thus we should choose $\dot{x}(t) = c(t, x(t), \rho(t, x(t))$ to obtain the system

$$\begin{cases} \dot{x}(t) = c(t, x(t), z(t)) \\ \dot{z}(t) = b(t, x(t), z(t)), \end{cases}$$

which we pair with the initial conditions $x(0) = \xi$ and $z(0) = f(\xi)$ for all possible ξ. It is clear that this situation can become very complicated, so let us just consider one simple example.

Example 15 (Damping)**.** *We consider*

$$\rho_t + \rho\rho_x = -a\rho, \quad a > 0.$$

The characteristic equations are

$$\dot{x}(t) = z(t) \quad \text{and} \quad \dot{z}(t) = -az(t).$$

With $x(0)=\xi$ *as before, we first solve*
$$z(t)=e^{-at}f(\xi)$$
and then obtain
$$x(t)=\xi+\tfrac{1-e^{-at}}{a}f(\xi).$$
We conclude that
$$\rho(t,x)=e^{-at}f(\xi),\quad \textit{where}\quad \xi=\xi(t,x)\quad \textit{satisfies}\quad x=\xi+[\tfrac{1-e^{-at}}{at}]tf(\xi).$$
The term $-a\rho$ *has a damping effect on the solution itself, but is it enough to prevent shock formation?*

Let us show that if f *is differentiable with* $f'(\xi)\geq -a$ *for all* ξ*, then characteristics cannot cross. Indeed, suppose there exists some* (t,x) *and* $\xi\neq\eta$ *such that*
$$x=\xi+[\tfrac{1-e^{-at}}{at}]tf(\xi)=\eta+[\tfrac{1-e^{-at}}{at}]tf(\eta).$$
Rearranging yields
$$\tfrac{f(\xi)-f(\eta)}{\xi-\eta}=-\tfrac{a}{1-e^{-at}}.$$
Thus by the mean value theorem, there exists x_0 *such that*
$$f'(x_0)=-\tfrac{a}{1-e^{-at}}<-a.$$

We generalize a bit further now and briefly discuss fully nonlinear first order equations of the form
$$H(y,\phi,\nabla\phi)=0$$
for some smooth function H. Here $\phi=\phi(y)$, and we write $y=(t,x)\in\mathbb{R}\times\mathbb{R}^n$. Then $\nabla\phi=(\partial_t\phi,\nabla_x\phi)$. We assume that we have an initial condition of the form
$$\phi(0,x)=f(x),\quad x\in\Gamma\subset\mathbb{R}^n.$$

Using the strategy introduced above (the *method of characteristics*), we will define a curve $y=y(s)$ and let $z(s)=\phi(y(s))$ and $p(s)=\nabla\phi(y(s))$. For any choice of y, one has
$$\dot z=p\cdot\dot y\quad\text{and}\quad \dot p=[\nabla^2\phi(y(s))]\dot y.$$
To find a good choice for $\dot y$, we start from the PDE and differentiate with respect to each y_i. Writing the arguments of H as $(y,z,p)\in\mathbb{R}^{n+1}\times\mathbb{R}\times\mathbb{R}^{n+1}$, this yields
$$\nabla_yH+\partial_zH\nabla\phi+\nabla^2\phi\nabla_pH=0.$$
Thus, if we define
$$\dot y(s)=\nabla_pH(y(s),z(s),p(s)),$$
we obtain
$$\dot p(s)=-\nabla_yH(y(s),z(s),p(s))-\partial_zH(y(s),z(s),p(s))p(s)$$

We are therefore led to the following ODE system (in $2n+3$ variables):
$$\begin{aligned}\dot y(s)&=\nabla_pH(y(s),z(s),p(s))\\ \dot p(s)&=-\nabla_yH(y(s),z(s),p(s))-\partial_zH(y(s),z(s),p(s))p(s),\\ \dot z(s)&=p(s)\cdot\dot y(s).\end{aligned}\tag{5.3}$$
For initial conditions, we fix $\xi\in\mathbb{R}^n$ and we must have that
$$y(0)=(0,\xi),\quad \pi[p(0)]=\nabla f(\xi),\quad z(0)=f(\xi),$$

where π is projection onto the last n components. This determines all but the first component of p, i.e. we need one more constraint. For this, we use the fact that the PDE must be satisfied. This requires

$$H(y(0), z(0), p(0)) = 0.$$

Together with the preceding display, this provides a set of $2n+3$ *compatibility conditions* for the initial data. Let us call a triple (y^0, z^0, p^0) satisfying all of these constraints *admissible*.

Now, given an admissible triple of data, we should be able to solve (5.3) (for reasonable H) by appealing to ODE existence results. To obtain an actual (local) solution from the ODE solutions, we need to know that (i) admissibility persists in a neighborhood of ξ and (ii) every point in some neighborhood of $(0,\xi)$ can be reached by some characteristic curve.

The key condition turns out to be the following:

Definition 1. *An admissible triple (y^0, z^0, p^0) is called* noncharacteristic *if*

$$\partial_{p_1} H(y^0, z^0, p^0) \neq 0.$$

Lemma 5.1. *Let (y^0, z^0, p^0) be a noncharacteristic, admissible triple with $y^0 = (0, \xi^0)$. Then for all ξ sufficiently close to ξ^0, there exists a (unique) noncharacteristic, admissible triple (y, z, p) with $y = (0, \xi)$.*

Proof. Let $q_0 = p_1^0$ (first component of p^0). Then, by assumption we have

$$H((0, \xi^0), f(\xi^0), (q_0, \nabla f(\xi^0))) = 0.$$

We wish to construct a function $q = q(\xi) \in \mathbb{R}$ (defined for ξ in a neighborhood of ξ^0) so that $q(\xi^0) = q_0$ and

$$H((0, \xi), f(\xi), (q(\xi), \nabla f(\xi))) \equiv 0. \tag{5.4}$$

We can do this using the implicit function theorem. Define the function $G : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$G(\xi, q) = H((0, \xi), f(\xi), (q, \nabla f(\xi)).$$

Then $G(\xi^0, q_0) = 0$ and the non-characteristic condition reads $\partial_q G(\xi^0, q_0) \neq 0$. Thus by the implicit function theorem there exists $q = q(\xi)$ for ξ in a neighborhood of ξ^0 such that $G(\xi, q(\xi)) \equiv 0$. This yields (5.4).

Moreover, since $\xi \mapsto q(\xi)$ is continuous, we can guarantee that the noncharacteristic condition continues to hold at points $((0, \xi), f(\xi), (q(\xi), \nabla f(\xi))$ (after possibly shrinking the neighborhood around ξ^0). The uniqueness claim also comes from the implicit function theorem. $\square$

Suppose (y^0, z^0, p^0) is noncharacteristic and admissible with $y^0 = (0, \xi^0)$. Using the lemma above, we can produce noncharacteristic, admissible data (y^1, z^1, p^1) from any ξ^1 sufficiently close to ξ^0. In particular, we take $y^1 = (0, \xi^1)$, $z^1 = f(\xi^1)$, and $p^1 = (q(\xi^1), \nabla f(\xi^1))$ (where $q(\cdot)$ is as in the lemma). We can then use ODE theory to solve (5.3) using this data, at least locally in s. We denote the dependence on the initial point ξ by writing $y(s;\xi)$, $p(s;\xi)$, and $z(s;\xi)$.

Lemma 5.2. *Let (y^0, z^0, p^0) be a noncharacteristic, admissible triple with $y^0 = (0, \xi^0)$. Then there exists an interval $I \ni 0$, a neighborhood $W \ni \xi^0$, and a neighborhood $V \ni (0, \xi^0)$ such that for any $Y \in V$ we have*

$$Y = y(s; \xi)$$

for some (unique) $\xi \in W$ and $s \in I$.

Proof. We have that $y^0 = (0, \xi^0) = y(0; \xi^0)$. Thus the result will follow from the inverse function theorem, provided we check that $\nabla_{s,y} y(0; \xi^0)$ is nondegenerate.

In fact, in general we have $(0, \xi) = y(0; \xi)$ (in the neighborhood guaranteed by the previous lemma). Moreover from (5.3) we have $\dot{y}(0, \xi^0) = \nabla_p H(y^0, z^0, p^0)$. Thus

$$\nabla_{s,y} y(0, \xi^0) = \begin{bmatrix} \partial_{p_1} H(y^0, z^0, p^0) & 0 \\ * & I_n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix and $*$ denotes the remaining components of $\nabla_p H(y^0, z^0, p^0)$. In particular, this matrix is nondegerate precisely due to the noncharacteristic condition. □

Using the characteristic ODE system, we can therefore obtain the following local existence theorem:

Theorem 5.1. *Suppose that (y^0, z^0, p^0) is an admissible, noncharacteristic triple, with $y^0 = (0, \xi^0)$. Then there exists a neighborhood $W \ni \xi^0$, a neighborhood $V \ni (0, \xi^0)$, and a solution $\phi = \phi(t, x)$ to $H(y, \phi, \nabla\phi) = 0$ for $(t, x) \in V$, with $\phi(0, x) = f(x)$ for $x \in W$.*

The proof amounts to solving the characteristic ODE system, defining $\phi(y) = z(y(s; \xi))$, where $y(s; \xi)$ is the unique characteristic curve containing the point y, and then verifying that the resulting function satisfies the PDE. The details are carried out in [2, p. 107].

We have focused on the case of specifying data at $t = 0$. More generally, one can specify data along some other hypersurface. The noncharacteristic condition is then expressed in terms of the normal derivative to this surface. See [2] for more details.

Let us return to the simpler case that we considered initially.

Example 16. *Let us consider an equation of the form*

$$\phi_t + c(\phi) \cdot \nabla_x \phi = 0,$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and $c(\rho) \in \mathbb{R}^n$. As usual we prescribe an initial condition of the form $\phi(0, x) = f(x)$. (This is the higher dimensional generalization of the examples worked out in detail above).

We may write the equation as

$$H(\phi, \nabla\phi) = 0, \quad H(z, p) = (1, c(z)) \cdot p.$$

The characteristic ODE system becomes

$$\dot{y} = (1, c(z)), \quad \dot{z} = p \cdot (1, c(z)) = 0,$$

where we have used the PDE itself in the equation for z. We have omitted the equation for $\dot{p}$ because it turns out not to be needed here.

In particular (just like before) if we set $y(t) = (t, x(t))$ with $\dot{x} = c(z)$ then we have $z \equiv z(0)$. The conclusion is then

$$\phi(t, x) = f(\xi), \quad \textit{where} \quad \xi = \xi(t, x) \quad \textit{satisfies} \quad x = \xi + c(f(\xi))t,$$

which is exactly what we derived before.

In this setting, the noncharacteristic condition is automatically satisfied. Thus the theorem above seems to guarantee the existence of solutions locally in time.

However, we have seen examples of data in which shocks form instantly, as well as data for which we had regions of spacetime not reached by any characteristic curves. This seems troubling, but in fact there is no contradiction: instantaneous shocks and rarefaction waves were only occurred in the case of discontinuous initial data, whereas the analysis above assumed all functions were quite regular!

6. First order hyperbolic systems

Reference: Whitham Chapter 5

This section was actually skipped in lecture, because I felt it ended up being a bit too superficial, but it is here if you want to read it.

In this section we will consider first order hyperbolic systems, although our presentation will be fairly brief. We will also stick mostly to the case of one space dimension.

The general quasi-linear first order system then takes the form

$$\sum_{j=1}^{n}\big[A_{ij}\tfrac{\partial u_j}{\partial t}+a_{ij}\tfrac{\partial u_j}{\partial x}\big]+b_i=0,\quad i=1,\dots,n,$$

where the unknown functions are $u_i=u_i(t,x)$ with $(t,x)\in\mathbb{R}\times\mathbb{R}$ and the functions A,a,b may be functions of $u=(u_1,\dots,u_n)$ and of (t,x). In what follows, we will use the Einstein summation notation, in which repeated indices are summed. In particular, the system above can be written

$$A_{ij}\tfrac{\partial u_j}{\partial t}+a_{ij}\tfrac{\partial u_j}{\partial x}+b_i=0,\quad i=1,\dots,n. \tag{6.1}$$

The main question we will try to address in this section is the following: when can we call the system above 'hyperbolic'? As we will see, the definition essentially corresponds to the assertion that for short times, the behavior at a point can only be influenced by sufficiently close points. (This in turn generally allows for local solvability via characteristics.)

We first investigate whether we can use (6.1) to get a relation involving the directional derivatives of all of the u_j in a single direction (α,β).

We begin by considering general linear combinations of the n equations in (6.1), which may be written

$$\ell_i\big[A_{ij}\tfrac{\partial u_j}{\partial t}+a_{ij}\tfrac{\partial u_j}{\partial x}\big]+\ell_i b_i=0$$

for some $\ell=\ell(t,x,u)$. We would like to choose ℓ so that this equation takes the form

$$m_j\big[\alpha\tfrac{\partial u_j}{\partial t}+\beta\tfrac{\partial u_j}{\partial x}\big]+\ell_j b_j=0 \tag{6.2}$$

for some (α,β).

Suppose for the moment that this is possible. We can then define a parametrized curve $(T(\eta),X(\eta))$ with

$$T'(\eta)=\alpha,\quad X'(\eta)=\beta.$$

If we define $Z(\eta)=u(T(\eta),X(\eta))$, then (6.2) becomes

$$m_j\dot{Z}_j+\ell_j b_j=0,$$

which is in *characteristic form* and can be integrated.

What does it take to achieve (6.2)? The conditions are

$$\ell_i A_{ij}=\alpha m_j\quad\text{and}\quad \ell_i a_{ij}=\beta m_j,$$

which leads to

$$\ell_i(\beta A_{ij} - \alpha a_{ij}) = 0.$$

A nontrivial solution ℓ exists if and only if

$$\det[\beta A_{ij} - \alpha a_{ij}] = 0,$$

which we read as a condition on the direction (α, β). In order to solve locally via characteristics, we would like to be able to find n characteristic directions. This will form the basis of the definition of a hyperbolic system.

Before giving the definition, we first observe that either A or a could be singular (in which case the x, t directions could be characteristic). In fact, both could be singular so that both of these directions are characteristic. On the other hand, in some cases in which both matrices are singular, the system becomes too degenerate. One way to prevent this overly degenerate situation is to require that some rotation of the coordinate axes leads to a new system with nonsingular matrices. Equivalently, we require

$$\exists(\lambda, \mu) \neq (0, 0) : \det[\lambda A_{ij} + \mu a_{ij}] \neq 0. \tag{6.3}$$

We can now introduce the definition of a hyperbolic system:

Definition 2. *A system* (6.1) *satisfying* (6.3) *is* hyperbolic *if there exist n independent vectors ℓ^k and nonzero (α^k, β^k) such that*

$$\ell_i^k[\beta^k A_{ij} + \alpha^k a_{ij}] = 0.$$

Note that the directions (α^k, β^k) may not be distinct. (If they are distinct, one calls the system *totally hyperbolic*.)

We consider several examples.

Example 17. *The case $A_{ij} = \delta_{ij}$ is an important special case, and indeed when A is nondegenerate the system can generally be reduced to this form. Note in this case that if $\alpha = 0$, then the condition above becomes $\beta\ell_i\delta_{ij} = 0$, which can only have a nontrivial solution if $\beta = 0$. Thus we may assume $\alpha \neq 0$, and in particular we can normalize to $\alpha = 1$. Thus characteristic curves are never in the direction of the x axis, and we can parametrize the curves as $(t, x(t))$ (similar to the scalar case).*

The characteristic condition for a direction $(1, \beta)$ then reads

$$\det[\beta I - a] = 0.$$

In particular, the characteristic velocities (i.e. the possible values of β) correspond to the eigenvalues of the matrix a, and the vectors ℓ are eigenvectors (of the transpose of a, actually).

From this, we deduce that the system is hyperbolic if a has n distinct real roots. In particular, this is the case whenever a is a real symmetric matrix.

Example 18. *The equation*

$$u_{tt} - \gamma u_{xx} = 0$$

can be written as a system by defining $v = u_x$ and $w = u_t$. In particular, the PDE is equivalent to the first order system

$$v_t - w_x = 0,$$
$$w_t - \gamma v_x = 0.$$

This corresponds to $A = I$ *and*

$$a = \begin{bmatrix} 0 & -1 \\ -\gamma & 0 \end{bmatrix}.$$

If $\gamma > 0$*, the eigenvalues are* $\pm\sqrt{\gamma}$*, whereas if* $\gamma < 0$ *the eigenvalues are pure imaginary. For* $\gamma > 0$ *we obtain the wave equation (which is hyperbolic), while for* $\gamma < 0$ *it is Laplace's equation (which is elliptic, not hyperbolic).*

Example 19. *Consider the damped wave equation*

$$u_{tt} - \gamma u_{xx} + u = 0,$$

with $\gamma > 0$*. If we define* $\varphi = u_t + \sqrt{\gamma}u_x$*, then this equation can be rewritten as the system*

$$\begin{aligned} \varphi_t - \sqrt{\gamma}\varphi_x + u &= 0, \\ u_t + \sqrt{\gamma}u - \varphi &= 0, \end{aligned}$$

which has nonsingular coefficients.

Example 20 (Gas dynamics)**.** *In compressible inviscid flow of gas, the velocity is denoted by* u*, the pressure by* p*, the density by* ρ*, and the entropy by* S*. The equations modeling the dynamics are*

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \tfrac{1}{\rho}p_x &= 0, \\ S_t + uS_x &= 0, \end{aligned}$$

with $p = p(\rho, S)$*. The characteristic equations can be worked out to*

$$\begin{aligned} &\tfrac{dp}{dt} \pm \rho a \tfrac{du}{dt} = 0 \quad on \quad \tfrac{dx}{dt} = u \pm a, \\ &\tfrac{dS}{dt} = 0 \quad on \quad \tfrac{dx}{dt} = u, \end{aligned}$$

where $a^2 = (\frac{\partial p}{\partial \rho})|_{S=const}$*.*

In these notes, we will not pursue hyperbolic systems any further, but let us mention a few topics that the interested reader could pursue further (see Whitham Section 5.5 and on).

- Characteristics carry information from the boundary into the region of interest. Abrupt changes on the boundary produce abrupt changes propagating on characteristics through those boundary points. An abrupt change may be modeled by a discontinuity in the derivative of u, and indeed discontinuities in derivatives can only occur on characteristics. If a system has no characteristics, one expects that discontinuities on the boundary will be immediately smoothed out in the solution. (On the other hand, existence of characteristics does not *guarantee* discontinuities.)
- An interesting example that can be worked out in detail involves systems admitting constant solutions, i.e. $u_j \equiv$ const. In this case (assuming $A_{ij} = \delta_{ij}$ as above) one can write the 'wavefront' in the form $x = X(t)$ and expand the solution in powers of $\xi := x - X(t)$. Doing the same for the functions a_{ij}, one can use the PDE system and start matching coefficients in order to compute the variation of the higher order derivatives. In some cases one can obtain wave breaking (so that discontinuities are introduced in the functions u_j themselves). See Section 5.6 in Whitham, where some

examples from river flow (shallow water waves, flood waves, tidal bores) are worked out. See also Section 5.8 for more information about shock waves in the system setting.

- Whitham also discusses systems in higher dimensions (i.e. $x \in \mathbb{R}^n$ with $n \geq 2$), cf. Section 5.9.

Finally, let us say a bit more about the classification of second order linear equations of the form

$$A_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + B_i \frac{\partial \varphi}{\partial x_i} + C\varphi = 0 \tag{6.4}$$

(with A_{ij} symmetric). As we saw above, sometimes such an equation can be converted to a first-order system. However, it may be convenient to work with this second-order form directly.

Consider the possibility that φ has a jump discontinuity in its second derivative across some surface $\{S(x) = 0\}$ (but φ and $\nabla\varphi$ are continuous). We can then introduce local coordinates based on this surface, deducing that $\frac{\partial^2 \varphi}{\partial S^2}$ is discontinuous while other second order derivatives are continuous (here this notation refers to normal derivative on the surface). Taking the difference of limits of (6.4) on the two sides of the surface leads to

$$A_{ij} \frac{\partial S}{\partial x_i} \frac{\partial S}{\partial x_j} \left[\frac{\partial^2 \varphi}{\partial S^2} \right] = 0,$$

where $[\cdot]$ denotes the value of the jump of the derivative across the surface. In order to have this jump be nonzero, we must then have

$$A_{ij} \frac{\partial S}{\partial x_i} \frac{\partial S}{\partial x_j} = 0. \tag{6.5}$$

Now, the quadratic form $\xi \mapsto A_{ij}\xi_i\xi_j$ can be reduced (at each point x) to

$$\xi \mapsto a_1 \xi_1^2 + \cdots + a_m \xi_m^2.$$

If the a_i are all the same sign, there is no solution to (6.5). We call the equation *elliptic* (at the point x). If some of the a_i are zero, we call the equation *parabolic*; the most typical case is that one a_i is zero and the rest have the same sign. If the a_i are nonzero but not all the same sign, we call the equation *hyperbolic*; the most typical case is that one a_i has a different sign than all of the others.

7. Linear dispersive equations, Fourier transform

Reference: Whitham Chapter 11.

Our next topic is *linear dispersive PDE*. We will follow Whitham and give the following practical definition of a dispersive equation: A linear equation is *dispersive* if it admits *plane wave solutions* of the form

$$\varphi(t, x) = A \exp\{ik \cdot x - i\omega t\}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

The parameters above have the following interpretation:

- $A \in \mathbb{C}$ is the *amplitude*,
- $k \in \mathbb{R}^d$ is the *wave number*,
- $\omega \in \mathbb{R}$ is the *frequency*.

To satisfy a (linear) PDE, the pair (ω, k) must satisfy a *dispersion relation* of the form

$$G(\omega, k) = 0, \tag{7.1}$$

where G is determined by the parameters of the PDE.

Example 21. *The function $\varphi(t,x)$ defined above solves the linear wave equation $\varphi_{tt} - \gamma^2 \Delta\varphi = 0$ if and only if*

$$G(\omega, k) := \omega^2 - \gamma^2 |k|^2 = 0.$$

In what follows, we make the underlying assumption that we can solve (7.1) via

$$\omega = W(k) \in \mathbb{R}.$$

(This assumption that $\omega \in \mathbb{R}$ is part of what makes the equation *dispersive*.)

Note that it is possible that we obtain multiple solutions in the dispersion relation. For example, we have the solutions $\omega = \pm\gamma|k|$ in Example 21. The general solution is built using the *superposition principle*.

If the PDE has real coefficients, it is a bit awkward to take complex-valued solutions. In this case we may take the solution

$$\operatorname{Re}\varphi(t,x) = |A|\cos(k \cdot x - \omega t + \eta), \quad \eta = \arg(A).$$

Before introducing several examples, let us define two important concepts related to dispersive equations, namely, *phase velocity* and *group velocity*.

The quantity

$$\theta = k \cdot x - \omega t$$

is called the *phase* of the solution. The phase surfaces $\{\theta = \text{const}\}$ are parallel planes. The value of θ determines the position on the cycle between a crest and trough.

Consider some phase plane, say $P = \{x \in \mathbb{R}^d : k \cdot x - \omega t = c\}$. This can be put into the standard plane form by writing

$$P = \{x \in \mathbb{R}^d : k \cdot [x - (c + \omega t)\tfrac{k}{|k|^2}] = 0\}.$$

From this form we can see that any particular phase surface moves with normal velocity $\frac{\omega}{|k|}$ in the direction $\frac{k}{|k|}$. Thus we define the **phase velocity** $c = c(k)$ by

$$c = \tfrac{\omega}{|k|}\tfrac{k}{|k|}, \quad \text{also written} \quad c = \tfrac{W(k)}{|k|}\tfrac{k}{|k|}.$$

Note that different wave numbers generally lead to different phase speeds—this is also part of what makes the equation *dispersive*. In what follows, we will build general solutions by taking (continuous) linear combinations of plane wave solutions. We will show that if we build such a 'wave packet' solution centered at frequency k, then the packet essentially travels with velocity $\nabla W(k)$. Thus we will define the **group velocity** by

$$\text{group velocity } = \nabla W(k).$$

In order for the equation to be 'dispersive', we want to exclude possibilities such as $W(k) = ak + b$. We therefore generally impose the additional condition

$$|\nabla^2 W(k)| \neq 0.$$

We summarize the discussion above in the following rough definition:

Definition 3. *A linear PDE is called* dispersive *if it admits solutions of the form $A\exp\{i(k\cdot x-\omega t)\}$, where (k,ω) satisfy some dispersion relation $G(\omega,k)=0$. We require that this equation can be solved by taking $\omega=W(k)$ for some $\omega\in\mathbb{R}$ (and note that it is permitted to have multiple solutions). We further require that $|\nabla^2 W(k)|\neq 0$.*

Given a dispersive equation with dispersion relation $\omega=W(k)$, we define the phase velocity by $\frac{W(k)}{|k|}\frac{k}{|k|}$ and the group velocity by $\nabla W(k)$.

Let's look at some examples (and non-examples):

Example 22 (Examples and non-examples)**.**

- *The linear Schrödinger equation*
$$i\partial_t u=-\Delta u$$
is dispersive. The dispersion relation is $G(\omega,k)=\omega-|k|^2=0$. The solution is $W(k)=|k|^2\in\mathbb{R}$, so that plane wave solutions have the form
$$A\exp\{i(k\cdot x-|k|^2 t)\}.$$
The phase velocity is k; the group velocity is $2k$. The Hessian of W is nondegenerate for all k.
- *The linear wave equation*
$$u_{tt}-\Delta u=0$$
has dispersion relation $G(\omega,k)=-\omega^2+|k|^2=0$, which has solutions $W(k)=\pm|k|$. Then plane wave solutions have the form
$$A\exp\{i(k\cdot x\pm|k|t)\}.$$
The phase velocity is $\pm\frac{k}{|k|}$ and the group velocity is $\pm\frac{k}{|k|}$. The Hessian is singular at $k=0$. This equation is not quite dispersive according to our definition above.
- *The closely-related Klein–Gordon equation is*
$$u_{tt}-\Delta u+m^2u=0,\quad m\neq 0.$$
The dispersion relation is $G(\omega,k)=-\omega^2+|k|^2+m^2=0$, which has solutions $W(k)=\pm\sqrt{|k|^2+m^2}$. Plane wave solutions have the form
$$A\exp\{i(k\cdot x\pm\sqrt{|k|^2+m^2}t)\}.$$
The phase velocity and group velocity are
$$\frac{k\sqrt{|k|^2+m^2}}{|k|^2}\quad \textit{and}\quad \frac{k}{\sqrt{|k|^2+m^2}},$$
respectively. In this case $\nabla^2 W$ is nondegenerate for all k.
- *The heat equation*
$$u_t=\Delta u$$
is not dispersive. If we look for solutions of the form
$$A\exp\{i(k\cdot x-\omega t)\},$$
then we find that we must have $\omega=-i|k|^2$, which is not real. The factor $e^{-i\omega t}$ would become $e^{-|k|^2t}$, which decays in time.

- *The Korteveg-de-Vries equation is*
$$u_t + u_{xxx} - 6uu_x = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}.$$
The linear part of this equation is the Airy equation
$$u_t + u_{xxx} = 0.$$
The dispersion relation for this equation is $\omega = W(k) = -k^3$. Plane wave solutions have the form
$$A\exp\{i(kx - k^3 t)\}$$
The phase velocity is $-k^2$ and the group velocity is $-3k^2$. Note that $W''(k)$ degenerates to zero as $k \to 0$.
- *For water waves, we have the following dispersion relation*
$$\omega^2 = g|k|\tanh(|k|h)\,\{1 + \tfrac{T}{\rho g}|k|^2\},$$
where h is the depth, g is the gravitational constant, ρ is the density, and T is the surface tension (Whitham's book [8] contains a derivation of this dispersion relation — see Ch.13 therein). There are several interesting limits that one can consider.

  First, in the regime $\frac{T}{\rho g}|k|^2 \ll 1$, the surface tension effects become negligible and one obtains the approximation
$$\omega^2 = g|k|\tanh(|k|h).$$
Using this dispersion relation leads to so-called gravity water waves. In this case the phase and group velocities are
$$c(k) = \sqrt{\tfrac{g}{|k|}\tanh(|k|h)} \quad \textit{and} \quad C(k) = \tfrac{1}{2}c(k)(1 + \tfrac{2|k|h}{\sinh(2|k|h)}).$$
(Here since the dispersion relation is radial, we are using the radial derivative of ω to obtain the group velocity.)

  In the limit $|k|h \to \infty$, we have
$$\omega \sim (g|k|)^{\frac{1}{2}}, \quad c(k) \sim (\tfrac{g}{|k|})^{\frac{1}{2}}, \quad C(k) \sim \tfrac{1}{2}(\tfrac{g}{|k|})^{\frac{1}{2}},$$
while in the limit $|k|h \to 0$ we have
$$\omega \sim (g|k|)^{\frac{1}{2}}|k|, \quad c(k) \sim (gh)^{\frac{1}{2}}, \quad C(k) \sim (gh)^{\frac{1}{2}}.$$
  In the other regime $\frac{T}{\rho g}|k|^2 \gg 1$, the surface tension effect dominates and we have the approximation
$$\omega^2 = \tfrac{T}{\rho}|k|^3\tanh(|k|h).$$
Using this dispersion relation leads to so-called capillary water waves. Then the phase and group velocities are
$$c(k) = (\tfrac{T}{\rho}|k|\tanh(|k|h))^{\frac{1}{2}} \quad \textit{and} \quad C(k) = \tfrac{3}{2}c(k)(1 + \tfrac{2|k|h}{3\sinh(2|k|h)}).$$
In the limit $|k|h \to \infty$ we have
$$\omega \sim (\tfrac{T}{\rho})^{\frac{1}{2}}|k|^{\frac{3}{2}}, \quad c(k) \sim (\tfrac{T}{\rho})^{\frac{1}{2}}|k|^{\frac{1}{2}}, \quad C(k) \sim \tfrac{3}{2}(\tfrac{T}{\rho})^{\frac{1}{2}}|k|^{\frac{1}{2}},$$
while in the limit $|k|h \to 0$ we have
$$\omega \sim (\tfrac{Th}{\rho})^{\frac{1}{2}}|k|^2, \quad c(k) \sim (\tfrac{Th}{\rho})^{\frac{1}{2}}|k|, \quad C(k) \sim 2(\tfrac{Th}{\rho})^{\frac{1}{2}}|k|^2.$$

Let us now discuss the construction of solutions to linear, dispersive equations. Suppose that we have a linear dispersive PDE with dispersion relation $\omega = W(k) \in \mathbb{R}$. For any $k \in \mathbb{R}^d$ and $A \in \mathbb{C}$, we can therefore obtain a plane wave solution

$$A \exp\{i(x \cdot k - tW(k))\}.$$

By linearity, we can build more general solutions by taking some smooth, localized function $A = A(k)$ and defining

$$u(t,x) = \int_{\mathbb{R}^d} A(k) e^{i(x\cdot k - tW(k))}\, dk.$$

Remark 7.1. *For $A \in L^1(\mathbb{R}^d)$, the function $u \in L^\infty_{t,x}(\mathbb{R} \times \mathbb{R}^d)$.*

At time $t = 0$, we have

$$u(0,x) = \int_{\mathbb{R}^d} A(k) e^{ix\cdot k}\, dk.$$

This shows that $A(\cdot)$ and $u(0,\cdot)$ are linked via the *Fourier transform*, which we now introduce.

7.1. **The Fourier transform.** This section gives a very brief review of the Fourier transform. If you have not studied the Fourier transform before, I encourage you to seek out a more comprehensive reference!

Given a function $\varphi : \mathbb{R}^d \to \mathbb{C}$, the *Fourier transform* of φ is the function $\hat{\varphi} : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\hat{\varphi}(k) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \varphi(x) e^{-ik\cdot x}\, dx.$$

We may also write $\hat{\varphi} = \mathcal{F}\varphi$. We can initially take the domain of $\mathcal{F}$ to be $L^1(\mathbb{R}^d)$, in which case we can verify that $\mathcal{F}(L^1(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d)$ (this is called the Riemann–Lebesgue lemma).

For $f \in L^1 \cap L^2$, one can prove that in fact $\hat{f} \in L^2$, with

$$\int_{\mathbb{R}^d} |\hat{f}(k)|^2\, dk = \int_{\mathbb{R}^d} |f(x)|^2\, dx.$$

This is called the Plancherel identity or Parseval's Theorem. Using this, one can prove that the Fourier transform extends to a bijective isometry on L^2, and moreover one has the *Fourier inversion formula*

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot k} \hat{f}(k)\, dk$$

and can further define $\mathcal{F}^{-1} : L^2 \to L^2$.

There are some other important general principles surrounding the Fourier transform. For example, we can say that the Fourier transform interchanges decay and regularity. This ultimately refers to identities such as the following:

$$\mathcal{F}[\nabla\varphi](k) = ik\hat{\varphi}(k),$$

which we will return to below.

Remark 7.2. *The identity above shows that ∇ is an example of a 'Fourier multiplier operator'. In general, a Fourier multiplier operator T has the property that $\mathcal{F}[Tf](\xi) = m(\xi)\hat{f}(\xi)$ for some function m, called the* multiplier *of T. We can also* *write $T = \mathcal{F}^{-1} m \mathcal{F}$. Noting that $\mathcal{F}[\Delta f](\xi) = -|\xi|^2 \hat{f}(\xi)$, we can further introduce notation such as*

$$e^{it\Delta} = \mathcal{F}^{-1} e^{-it|\xi|^2} \mathcal{F}.$$

One can check that $e^{it\Delta} f$ is the solution to the linear Schrödinger equation with initial data f, that $e^{-t\partial_x^3} f$ is the solution to the Airy equation with data f, and so on.

Remark 7.3. *Fourier multiplier operators may also be expressed as* convolution *operators, namely: if $T = \mathcal{F}^{-1} m \mathcal{F}$, then*

$$Tf = (2\pi)^{-\frac{d}{2}} [\mathcal{F}^{-1} m] * f,$$

where the convolution $*$ is defined by

$$f * g(x) := \int f(x-y) g(y)\, dy.$$

This follows from the following convolution identity for the Fourier transform:

$$\mathcal{F}[f * g] = (2\pi)^{\frac{d}{2}} \hat{f}\, \hat{g}.$$

Now, given an initial condition $\varphi(x)$ and a linear dispersive PDE with dispersion relation $\omega = W(k)$, we see that the solution is given by

$$\begin{aligned} u(t,x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i(x\cdot k - tW(k))} \hat{\varphi}(k)\, dk \\ &= (2\pi)^{-d} \iint e^{i[(x-y)\cdot k - tW(k)]} \varphi(y)\, dy\, dk. \end{aligned}$$

We can also see from the identity above that

$$\hat{u}(t,k) = e^{-itW(k)} \hat{\varphi}(k).$$

Since $W(k) \in \mathbb{R}$, this implies $|\hat{u}(t,k)| \equiv |\hat{\varphi}(k)|$ for all t. This, in turn, guarantees conservation of all of the L^2-based Sobolev norms, which often have physical interpretations (e.g. mass and energy).

7.2. **Group velocity.** In this section we continue to assume that we are working with some linear, dispersive equation with some dispersion relation $\omega = W(k)$. We want to understand the statement that an initial wave-packet centered at frequency $k_0 \in \mathbb{R}^d$ travels with the group velocity $\nabla W(k_0)$. We will continue to assume that $\nabla^2 W$ is nondegenerate.

We will also see that even an initially localized wave packet will eventually disperse over time, with a decay rate determined by the dispersion relation. This pointwise decay coupled with the conservation of L^2-based Sobolev norms is characteristic linear, dispersive PDE.

To simplify the discussion, let us restrict attention to the case $d = 1$. We consider a solution of the form

$$\varphi(t,x) = \int F(k) e^{-i\chi t}\, dk, \quad \text{where} \quad \chi = \chi(t,x;k) = W(k) - k\tfrac{x}{t},$$

where we assume F and all of its derivatives are integrable. To say we have a wave packet localized at $k = k_0$ simply means that F is sharply localized around k_0. We write $F(k) \approx F(k_0)$.

At time $t = 0$ we have a wave packet at spatial position $x = 0$ and frequency $k = k_0$. Our question is: where should we expect to find the 'bulk' of the solution at some time $t \geq 1$? We are claiming that the solution will be found at $x \approx W'(k_0)t$.

To see this, we first observe that

$$\chi'(k) = W'(k) - \tfrac{x}{t} \approx W'(k_0) - \tfrac{x}{t}$$

on the support of F. Suppose first that (t, x) are such that $|W'(k_0) - \frac{x}{t}| \gtrsim 1$. Let us show that we can obtain arbitrary good decay for $\varphi(t, x)$ at such points.

We begin by writing

$$e^{-i\chi t} = \tfrac{i}{\chi' t} \tfrac{d}{dk} e^{-i\chi t}.$$

Then, integrating by parts, we have

$$\varphi(t, x) = \tfrac{1}{it} \int \tfrac{1}{\chi'} F'(k) e^{-i\chi t}\, dk.$$

As we are assuming $|\chi'(k; t, x)| \gtrsim 1$, we obtain the bound $|\varphi(t, x)| \lesssim \frac{1}{|t|}$. Repeating the argument once more, we can in fact obtain $|\varphi(t, x)| \lesssim \frac{1}{|t|^\ell}$ for any ℓ. Thus:

$$|\tfrac{x}{t} - W'(k_0)| \gtrsim 1 \implies |\varphi(t, x)| \lesssim_\ell \langle t\rangle^{-\ell} \quad \text{for any} \quad \ell \geq 1.$$

(This argument is known as the *nonstationary phase* argument.)

It remains to see that there actually *is* something to be found around $x = W'(k_0)t$ (the *stationary phase* point). At this point we have $\chi'(k_0) = 0$, so that we may write

$$\chi(k) \approx \chi(k_0) + \tfrac{1}{2}(k - k_0)^2 \chi''(k_0),$$

where we recall the overarching assumption that $\chi''(k_0) \neq 0$. Then

$$\varphi(t, x) \approx F(k_0) e^{-i\chi(k_0)t} \int e^{-\frac{i}{2}(k-k_0)^2 \chi''(k_0)t}\, dk.$$

This final integral can be computed explicitly[1]. The result is

$$\varphi(t, x) \approx \big(\tfrac{2\pi}{t|\chi''(k_0)|}\big)^{\frac{1}{2}} F(k_0) e^{-i\chi(k_0)t} \exp\{\tfrac{i\pi}{4}\operatorname{sgn}(\chi''(k_0))\}, \quad x \approx W'(k_0)t.$$

Thus the wave packet propagates with velocity $W'(k_0)$, decaying over time at a rate of $t^{-1/2}$.

Similar computations go through in $\mathbb{R}^d$, with characteristic decay rate of $t^{-d/2}$. Recalling that the L^2-norm is conserved, we see that the characteristic width of the solution must also grow at a rate of $\sim t$.

The argument above breaks down if $W''(k)$ becomes nondegenerate, and generally one obtains different (weaker) decay rates. For example, while the Schrödinger equation and Klein–Gordon equation have decay rates of $t^{-\frac{d}{2}}$, the Airy equation has the decay rate $t^{-1/3}$, and the wave equation exhibits a decay rate essentially like $t^{-(d-1)/2}$. Thus, for example, we have:

- $\|e^{it\Delta}\varphi\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|\varphi\|_{L_x^1(\mathbb{R}^d)}$,
- $\|e^{-t\partial_x^3}\varphi\|_{L_x^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{3}} \|\varphi\|_{L_x^1(\mathbb{R})}$.

The dispersive estimate for the wave equation is a bit more complicated to state, and we skip it here.

The dispersive estimates, together with L^2 conservation, can be used together with other tools from harmonic analysis to prove so-called *Strichartz estimates* for

[1] This relies on the Gaussian integral $\int_{\mathbb{R}} e^{-\alpha z^2}\, dz = (\frac{\pi}{\alpha})^{\frac{1}{2}}$.

a variety of dispersive equations. Such estimates are then essential in proving local well-posedness for nonlinear dispersive equations.

Let us also consider some slightly more general situations, e.g. variable-coefficient equations (which cannot be treated so simply using the Fourier transform). Let us restrict to one spatial dimension to keep things simple.

In the constant coefficient case, we have the phase

$$\theta(t,x) = k \cdot x - \omega t, \quad \omega = W(k).$$

We can then recognize the wave number $k = \partial_x \theta$ and the frequency $\omega = -\partial_t \theta$. In the non-constant coefficient case, we may still have a dispersion relation $\omega = W(k)$, and we may be able to construct slowly varying 'wavetrains' of the form $A(t,x)\exp\{i\theta(t,x)\}$. We can then define a local wave number and frequency b y

$$k = k(t,x) = \partial_x \theta(t,x) \quad \text{and} \quad \omega = -\partial_t \theta(t,x).$$

Using the dispersion relation, we then obtain

$$\partial_t k + \partial_x \omega = 0, \quad \text{so that} \quad \partial_t k + W'(k)\partial_x k = 0.$$

This is a $1d$ conservation law, like the type we studied above! Here k is the density of waves, ω is the flux of waves, the equation is conservation of waves, and evidently $W'(k)$ is the propagation velocity for wave number k.

Let us illustrate this idea with a simple example, namely, the linear Schrödinger equation with an external potential.

Example 23. *Consider the* 1d *Schrödinger equation*

$$iu_t = -u_{xx} + Vu,$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$ *and* $V : \mathbb{R} \to \mathbb{R}$. *In order to obtain a plane wave type solution, we look for a solution of the form*

$$u(t,x) = A(t,x)\exp\{i\theta(t,x)\}.$$

Then a direct computation yields

$$\begin{aligned} &iu_t + u_{xx} - Vu \\ &\quad = e^{i\theta}\{A_{xx} - (\theta_x)^2 A - A\theta_t - VA + i(A_t + 2\theta_x A_x + \theta_{xx})\}. \end{aligned}$$

Thus we obtain a solution if we have

$$\begin{aligned} A_t + 2\theta_x A_x + A\theta_{xx} &= 0, \\ \theta_t + (\theta_x)^2 &= V - \tfrac{A_{xx}}{A}. \end{aligned}$$

As we want a 'slowly varying wavetrain', let us simply set the A_{xx} *term equal to zero in the equation for* θ. *Then we get the following equation for the phase:*

$$\theta_t + (\theta_x)^2 = V.$$

If $V \equiv 0$, *we can recover the phase for the free linear Schrödinger equation, namely* $\theta(t,x) = kx - k^2 t$. *In the general case, we use the method of characteristics. We begin by differentiating the equation with respect to* x *to observe*

$$(\theta_x)_t + 2\theta_x(\theta_x)_x = V_x.$$

Now let us introduce a curve $x(t)$ *and define* $p(t) = \theta_x(t, x(t))$. *Then*

$$\dot{p} = (\theta_x)_t + \dot{x}(\theta_x)_x,$$

so that if we pick $\dot{x} = 2p$ *we arrive at the ODE system*

$$\dot{x}(t) = 2p(t), \quad \dot{p}(t) = V_x(x(t)).$$

If we solve this system, then we can also compute θ_t *along the curve* $x(t)$ *by using the original PDE:*

$$\partial_t \theta(t, x(t)) = \theta_t(t, x(t)) + \dot{x}\theta_x(t, x(t)) = V(x(t)) + p(t)^2.$$

The local wave number and frequency along the curve are given by

$$k = \partial_x \theta(t, x(t)) = p(t), \quad \omega = -\partial_t \theta(t, x) = -p(t)^2 - V(x(t)),$$

where $\dot{x}(t) = 2p(t)$. *This can be read as a change in the frequency for a given wave number in a way that depends explicitly on the potential.*

7.3. **A simple well-posedness result.** Let us close this section with a simple existence result for the $1d$ nonlinear Schrödinger equation:

$$iu_t + u_{xx} = -|u|^2 u. \tag{7.2}$$

Theorem 7.1. *For any* $u_0 \in H^1(\mathbb{R})$, *there exists a unique solution* u *to* (7.2) *with* $u|_{t=0} \in u_0$ *on some time interval* $[0, T]$, *where* $T = T(\|u_0\|_{H^1})$.

We will need a 'Sobolev embedding' estimate:

Lemma 7.1. *Let* $u \in H^1(\mathbb{R})$. *Then*

$$\|u\|_{L_x^\infty(\mathbb{R})} \lesssim \|u\|_{L_x^2(\mathbb{R})}^{\frac{1}{2}} \|\partial_x u\|_{L_x^2(\mathbb{R})}^{\frac{1}{2}}.$$

Proof. Let $x \in \mathbb{R}$. By the fundamental theorem of calculus and the Cauchy–Schwarz inequality,

$$|u(x)|^2 = \left| \int_x^\infty 2u(y)u'(y)\, dy \right| \leq 2\|u\|_{L^2} \|u'\|_{L^2}.$$

As the estimate holds uniformly in x, the result follows. □

Proof of Theorem 7.1. We will use the Duhamel formulation of (7.2). This means that we will construct a solution to the following equation

$$u(t) = [\Phi u](t) := e^{it\Delta} u_0 + i \int_0^t e^{i(t-s)\Delta} [|u|^2 u(s)]\, ds,$$

where

$$e^{it\Delta} = \mathcal{F}^{-1} e^{-it|\xi|^2} \mathcal{F}.$$

The strategy for accomplishing this will be to show that the operator Φ is a contraction on a suitable complete metric space. To this end, we let $T > 0$ be a small parameter to be determined below, and we define a space X consisting of functions $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfying the bounds

$$\|u(t)\|_{L_x^2} \leq 2M \quad \text{and} \quad \|\partial_x u(t)\|_{L_x^2} \leq 2M \quad \text{for all} \quad t \in [0, T],$$

where $M = \|u_0\|_{H_x^1}$. We equip this space with the metric

$$d(u, v) = \sup_{t \in [0, T]} \|u(t) - v(t)\|_{L_x^2}.$$

The interested reader should verify that (X, d) is a complete metric space.

We now wish to prove that:

(i) $\Phi : X \to X$ and
(ii) $d(\Phi u, \Phi v) \leq \frac{1}{2} d(u, v)$ for any $u, v \in X$.

First, let $u \in X$. Then

$$\begin{aligned}
\|\Phi u\|_{L_x^2} &\leq \|u_0\|_{L_x^2} + \int_0^t \||u|^2 u\|_{L_x^2}\, ds \\
&\leq M + \int_0^t \|u\|_{L_x^\infty}^2 \|u\|_{L_x^2}\, ds \\
&\leq M + \int_0^t \|u\|_{H_x^1}^3\, ds \\
&\leq M + T(2C)^3 M^3.
\end{aligned}$$

Thus for $T = T(M)$ sufficiently small, we have $\|\Phi u\|_{L_x^2} \leq 2M$ for all $t \in [0,T]$. Similarly, using the chain rule,

$$\begin{aligned}
\|\partial_x \Phi u\|_{L_x^2} &\leq \|\partial_x u_0\|_{L_x^2} + \int_0^t \||u|^2 \partial_x u\|_{L_x^2}\, ds \\
&\leq M + \int_0^t \|u\|_{L_x^\infty}^2 \|u\|_{L_x^2}\, ds \\
&\leq M + T(2C)^3 M^3,
\end{aligned}$$

so that if $T = T(M)$ is sufficiently small, we obtain $\|\partial_x \Phi u\|_{L_x^2} \leq 2M$ for all $t \in [0,T]$. We conclude that $\Phi : X \to X$ (if T is small enough).

Now let $u, v \in X$. Using the pointwise estimate

$$\big||u|^2 u - |v|^2 v\big| \lesssim (|u|^2 + |v|^2)|u - v|,$$

we obtain

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{L_x^2} &\leq \int_0^t [\|u\|_{L_x^\infty}^2 + \|v\|_{L_x^\infty}^2 \|u - v\|_{L_x^2} \\
&\lesssim TM^2 \|u - v\|_{L_t^\infty L_x^2},
\end{aligned}$$

which (for T sufficiently small) implies (ii) above.

We conclude from the contraction mapping principle that there exists a unique $u \in X$ such that $u = \Phi(u)$. $\square$

8. Korteveg de Vries Equation

Reference: Whitham Chapter 16

In this section we study the Korteveg de Vries equation

$$u_t + u_{xxx} - 6uu_x = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{8.1}$$

This is a nonlinear dispersive PDE modeling small amplitude water waves.

Our goal is to present some type of existence result that doesn't require a ton of analysis background. In particular we will prove the existence of solutions with sufficiently regular initial data. In particular, we will consider $u|_{t=0} = u_0 \in H^2(\mathbb{R})$, where H^2 is the *Sobolev space* defined by the norm

$$\|f\|_{H^2}^2 = \|f\|_{L^2}^2 + \|\Delta f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 + \|\xi^2 \hat{f}\|_{L^2}^2.$$

This is far from optimal. In fact, the optimal result is for $u_0 \in H^{-1}(\mathbb{R})$, due to Killip and Visan [5].

Theorem 8.1 (Local well-posedness)**.** *For any $u_0 \in H^2(\mathbb{R})$, there exists a (weak) solution to* (8.1) *with $u|_{t=0} = u_0$ on some interval $[0,T]$, where $T = T(\|u_0\|_{H^2})$.*

The argument used to prove Theorem 7.1 breaks down in this setting due to the presence of the derivative in the nonlinearity.

We will instead use the method of 'parabolic regularization' and energy estimates. We introduce a parameter $0 < \varepsilon \ll 1$ and consider the family of PDE

$$u_t + u_{xxx} - 6uu_x = -\varepsilon u_{xxxx}, \quad u|_{t=0} = u_0. \tag{8.2}$$

The regularizing fourth order term yields the following smoothing estimate, which will be essential for handling the derivative in the nonlinearity:

Lemma 8.1 (Smoothing estimate)**.** *For any $f \in L^2$,*

$$\|\partial_x e^{-t(\partial_x^3+\varepsilon\partial_x^4)} f\|_{L_x^\infty} \lesssim (\varepsilon t)^{-\frac{3}{8}} \|f\|_{L_x^2}.$$

Proof. As $e^{-t\partial_x^3}$ preserves the L^2-norm, it is enough to prove the estimate

$$\|e^{-\varepsilon t\partial_x^4}\partial_x f\|_{L_x^\infty} \lesssim (\varepsilon t)^{-\frac{3}{8}} \|f\|_{L_x^2}.$$

We begin by writing

$$e^{-\varepsilon t\partial_x^4}\partial_x f = (2\pi)^{-\frac{d}{2}}\mathcal{F}^{-1}[ike^{-\varepsilon tk^4}] * f.$$

We now use the convolution inequality

$$\|g * f\|_{L^\infty} \lesssim \|g\|_{L^2}\|f\|_{L^2},$$

and the problem now follows from observing that

$$\|\mathcal{F}^{-1}[ike^{-\varepsilon tk^4}]\|_{L^2} = \|ke^{-\varepsilon tk^4}\|_{L_k^2} \lesssim (\varepsilon t)^{-\frac{3}{8}}$$

(a consequence of Plancherel and a change of variables). □

We now prove a basic existence result for (8.2).

Proposition 8.1 (Short-time existence for the regularized equation)**.** *For any $u_0 \in L^2$ and $0 < \varepsilon \ll 1$, there exists a solution u to* (8.2) *with $u|_{t=0} = u_0$ on some time interval $[0, \delta]$, where $\delta = \delta(\|u_0\|_{L^2}, \varepsilon)$.*

Proof. We will use the Duhamel formulation of (8.2). This means that we will construct a solution to the following equation

$$u(t) = [\Phi u](t) := e^{-t\mathcal{L}}u_0 + 6\int_0^t e^{-(t-s)\mathcal{L}}[u(s)u_x(s)]\,ds,$$

where

$$\mathcal{L} := \partial_x^3 + \varepsilon\partial_x^4.$$

The strategy for accomplishing this will be to show that the operator Φ is a contraction on a suitable complete metric space. To this end, we let $\delta > 0$ be a small parameter to be determined below, and we define a space X consisting of functions $u : [0, \delta] \times \mathbb{R} \to \mathbb{R}$ satisfying the bounds

$$\|u(t)\|_{L_x^2} \leq 2M \quad \text{and} \quad \|\partial_x u(t)\|_{L_x^\infty} \leq 2CM(\varepsilon t)^{-\frac{3}{8}} \quad \text{for all} \quad t \in [0, \delta],$$

where $M = \|u_0\|_{L_x^2}$. We equip this space with the metric

$$d(u, v) = \sup_{t\in[0,\delta]} \big\{ \|u(t) - v(t)\|_{L_x^2} + (\varepsilon t)^{\frac{3}{8}} \|u_x(t) - v_x(t)\|_{L_x^\infty} \big\}.$$

The interested reader should verify that (X, d) is a complete metric space.

We now endeavor to prove that:

(i) $\Phi : X \to X$ and

(ii) $d(\Phi u, \Phi v) \leq \frac{1}{2} d(u, v)$ for any $u, v \in X$.

We take $u \in X$. Then (noting that $e^{-t\mathcal{L}}$ maps L^2 to L^2 boundedly)

$$\begin{aligned}
\|\Phi u\|_{L^2_x} &\leq \|u_0\|_{L^2_x} + C \int_0^t \|u u_x\|_{L^2_x}\, ds \\
&\leq M + C \int_0^t \|u\|_{L^2} \|\partial_x u\|_{L^\infty_x}\, ds \\
&\leq M + \int_0^t 4CM^2 (\varepsilon s)^{-\frac{3}{8}}\, ds \\
&\leq M + 4CM^2 \varepsilon^{-\frac{3}{8}} \delta^{\frac{5}{8}}.
\end{aligned}$$

Thus

$$\|\Phi u\|_{L^2_x} \leq 2M$$

provided $\delta = \delta(\varepsilon, M)$ is sufficiently small.

Next,

$$\begin{aligned}
\|\partial_x \Phi u\|_{L^\infty_x} &\leq CM(\varepsilon t)^{-\frac{3}{8}} + C\varepsilon^{-\frac{3}{8}} \int_0^t |t-s|^{-\frac{3}{8}} \|u_x u\|_{L^2_x}\, ds \\
&\leq CM(\varepsilon t)^{-\frac{3}{8}} + C\varepsilon^{-\frac{3}{8}} \int_0^t |t-s|^{-\frac{3}{8}} \|u_x\|_{L^\infty_x} \|u\|_{L^2_x}\, ds \\
&\leq CM(\varepsilon t)^{-\frac{3}{8}} + CM^2 \varepsilon^{-\frac{3}{4}} \int_0^t |t-s|^{-\frac{3}{8}} |s|^{-\frac{3}{8}}\, ds \\
&\leq CM(\varepsilon t)^{-\frac{3}{8}} + CM^2 \varepsilon^{-\frac{3}{4}} t^{-\frac{3}{8}} \delta^{\frac{5}{8}},
\end{aligned}$$

where we have estimated

$$\sup_{t \in [0,\delta]} |t|^{\frac{3}{8}} \int_0^t |t-s|^{-\frac{3}{8}} |s|^{-\frac{3}{8}}\, ds \lesssim \delta^{\frac{5}{8}}$$

by splitting the integral into $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$ and estimating each piece separately. By choosing $\delta = \delta(\varepsilon, M)$ sufficiently small, we therefore obtain

$$\|\partial_x \Phi u\|_{L^\infty_x} \leq 2CM(\varepsilon t)^{-\frac{3}{8}}$$

for all $t \in [0, \delta]$.

Thus we have shown that for $\delta = \delta(\varepsilon, M)$ sufficiently small, we have $\Phi : X \to X$.

Now we estimate similarly to prove the contraction property. We write

$$u u_x - v v_x = u(u_x - v_x) + v_x(u - v)$$

and (given $u, v \in X$) estimate

$$\begin{aligned}
\|u(t) - v(t)\|_{L^2_x} &\leq \int_0^t \left\{ \|u\|_{L^2_x} \|u_x - v_x\|_{L^\infty_x} + \|v_x\|_{L^\infty_x} \|u - v\|_{L^2_x} \right\} ds \\
&\lesssim \int_0^t M(\varepsilon s)^{-\frac{3}{8}} d(u, v)\, ds \\
&\lesssim M \varepsilon^{-\frac{3}{8}} \delta^{\frac{5}{8}} d(u, v).
\end{aligned}$$

Similarly,

$$\begin{aligned}\|\partial_x u(t)-\partial_x v(t)\|_{L^2_x} &\lesssim M\varepsilon^{-\frac34}\int_0^t |t-s|^{-\frac38}|s|^{-\frac38}d(u,v)\,ds\\ &\lesssim M\varepsilon^{-\frac34}\delta^{\frac58}|t|^{-\frac38}d(u,v).\end{aligned}$$

This, for $\delta=\delta(\varepsilon,M)$ sufficiently small, we can obtain

$$d(\Phi u,\Phi v)\leq \tfrac12 d(u,v).$$

Thus by the contraction mapping princple, there exists unique $u\in X$ such that $u=\Phi(u)$ (provided $\delta=\delta(\varepsilon,M)$ is taken small enough). □

By iterating the result above, one can obtain global existence to (8.2) for data in L^2 (forward in time).

Corollary 8.1. *For any $\varepsilon>0$ and any $u_0\in L^2$, there exists a unique (forward) global solution to* (8.2) *with $u|_{t=0}=u_0$.*

Proof. The key observation is the following: if u solves (8.2) with $\varepsilon\geq 0$, then

$$\|u(t)\|_{L^2_x}\leq \|u(0)\|_{L^2_x}$$

for all $t\geq 0$. With this fact in hand, we can iterate the local existence result above to obtain a global-in-time solution. For the proof of this fact, we observe that if u solves (8.2), then (using integration by parts)

$$\begin{aligned}\tfrac{d}{dt}\int u^2\,dx &= 2\int uu_t\,dx\\ &= 2\int[-uu_{xxx}-\varepsilon uu_{xxxx}+6u^2u_x]\,dx\\ &= 2\int u_xu_{xx}\,dx-2\varepsilon\int u_{xx}u_{xx}\,dx+4\int\partial_x(u^3)\,dx\\ &= \int\partial_x(u_x^2)\,dx+4\int\partial_x(u^3)\,dx-2\varepsilon\int u_{xx}u_{xx}\,dx.\end{aligned}$$

The first two terms are zero, while the final term is ≤ 0. Thus the claim follows. □

The next step of the argument is to show that if $u_0\in H^2$, then the solution to (8.2) obeys H^2 bounds that are uniform in t *and* in $\varepsilon>0$. This will be essential in 'sending $\varepsilon\to 0$' to obtain a solution to (8.1).

Lemma 8.2 (Energy estimate)**.** *If u solves* (8.2) *with $u|_{t=0}=u_0\in H^2$, then*

$$\sup_{t\in[0,T]}\|u(t)\|_{H^2}^2\lesssim \|u_0\|_{H^2}^2,$$

where $T=\mathcal{O}(\|u_0\|_{H^2}^{-1})$.

Proof. Let u be a solution to (8.2) with $u|_{t=0}=u_0\in H^2$. We will perform a formal computation of the time derivative of the quantity

$$E(t):=\int u^2+u_x^2+u_{xx}^2\,dt$$

using (8.2). We already computed above that

$$\tfrac{d}{dt}\int u^2=-2\varepsilon\int u_{xx}^2\,dx.$$

We next compute

$$\begin{aligned}\tfrac{d}{dt}\int u_x^2\,dx &= \int u_x u_{tx}\,dx\\ &= -\int u_{xx}[-u_{xxx}-\varepsilon u_{xxxx}+6uu_x]\,dx\\ &= \int \partial_x[\tfrac12 u_{xx}^2] - 2\varepsilon\int u_{xxx}^2\,dx - 12\int uu_xu_{xx}\,dx.\end{aligned}$$

The first term vanishes, while the second term is ≤ 0. The final term can be estimated in magnitude (using Cauchy–Schwarz and Lemma 7.1) by

$$\|u_x\|_{L_x^2}\|u_{xx}\|_{L_x^2}\|u\|_{L_x^\infty} \lesssim \|u\|_{H_x^2}^3 \lesssim [E(t)]^{\frac32}.$$

Finally,

$$\begin{aligned}\tfrac{d}{dt}\int u_{xx}^2\,dx &= \int u_{xx}u_{txx}\,dx\\ &= \int u_{xxxx}[-u_{xxx}-\varepsilon u_{xxxx}+6uu_x]\,dx\\ &= \int -\tfrac{d}{dx}[\tfrac12 u_{xxx}^2] - \varepsilon\int u_{xxxx}^2\,dx + 6\int u_{xxxx}uu_x\,dx.\end{aligned}$$

The first term vanishes, while the second term is ≤ 0. We rewrite the final term as follows:

$$\begin{aligned}\int u_{xxxx}uu_x\,dx &= \int u_{xx}[3u_xxu_x + uu_{xx}]\,dx\\ &= \int 3u_{xx}^2u_x + \int u\partial_x(\tfrac12 u_{xx}^2)\,dx\\ &= \int \tfrac52 u_{xx}^2u_x\,dx.\end{aligned}$$

Using Cauchy–Schwarz and Lemma 7.1, this term is estimated by

$$\|u_{xx}\|_{L_x^2}^2\|u_x\|_{L_x^\infty} \lesssim \|u\|_{H_x^2}^3 \lesssim [E(t)]^{\frac32}.$$

It follows that

$$\tfrac{d}{dt}E(t) \leq 2C[E(t)]^{\frac32} \quad \text{for} \quad t\geq 0.$$

Rearranging, this yields

$$\tfrac{d}{dt}[E^{-\frac12}] \geq -C$$

and so

$$[E(t)]^{-\frac12} \geq [E(0)]^{-\frac12} - Ct.$$

Thus

$$E(t) \leq \left[\frac{1}{E(0)^{-1/2}-Ct}\right]^2.$$

In particular, we obtain $E(t)\lesssim E(0)$ for $t \ll [E(0)]^{-\frac12}$, which yields the desired bound. $\square$

Let us now turn to the proof of Theorem 8.1.

Proof of Theorem 8.1. Given $u_0 \in H^2$, we write u^ε for the solution to (8.2) with initial data $u^\varepsilon|_{t=0} = u_0$. We let T be as in Lemma 8.2. We now prove that we can obtain a limit u in $L_t^\infty H_x^1([0,T]\times\mathbb{R})$ as $\varepsilon \to 0$, and that this limit satisfies (8.1), at least in a weak sense. The key ingredient is the uniform (in t and ε) bounds in H^2 for the solutions u^ε on $[0,T]$.

Note that by using the equation, these bounds imply that $\partial_t u^\varepsilon \in L_t^\infty H_x^{-2}$ with uniform bounds. (This is straightforward to check; it is helpful to write $6uu_x = 3\partial_x(u^2)$.)

Now, on any compact subset $K \subset \mathbb{R}$, the embedding $H^2(K) \hookrightarrow H^1(K) \hookrightarrow H^{-2}(K)$ is compact. This is an analysis fact that we will take for granted here. Now, as

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{L_t^2 H_x^2([0,T]\times K)} \lesssim 1 \quad \text{and} \quad \sup_{\varepsilon>0} \|\partial_t u^\varepsilon\|_{L_t^1 H_x^{-2}([0,T]\times K)} \lesssim 1,$$

we can use the Aubin–Lions–Simons lemma (another analysis result we take for granted, see Lemma 2.1) to obtain that u^ε converges to some limit u (along a sequence in ε) strongly in $L_t^2 H_x^1([0,T]\times K)$. By using a diagonal argument, we can upgrade this to $L_t^2 H_{x,\mathrm{loc}}^1$.

We now wish to upgrade this convergence to $L_t^2 H_x^1$ (removing the local restriction). The key is to observe that one can prove a localized version of the energy estimates established above. For example if we let χ_R be a smooth cutoff to $\{|x| \le R\}$ with $R \gg 1$, we can use computations similar to ones carried out above to obtain

$$\partial_t \int \chi_R[u^2 + u_x^2]\,dx \lesssim R^{-1}\{\|u\|_{L_t^\infty H_x^2}^2 + \|u\|_{L_t^\infty H_x^2}^3\}$$

for solutions to (8.2). This implies that for any $\delta > 0$, there exists R sufficiently large (depending on δ and u_0 but *not* on ε) that

$$\|u^\varepsilon\|_{L_t^\infty H_x^1(|x|>R)} \le \delta$$

on $[0,T]$, uniformly in $\varepsilon > 0$. (The details are left as an exercise to the interested reader.)

Then given $\delta > 0$ and $u^{\varepsilon_1}, u^{\varepsilon_2}$, we can choose $R = R(u_0,\delta)$ large and estimate

$$\begin{aligned} \|u^{\varepsilon_1} - u^{\varepsilon_2}\|_{L_t^2 H_x^1} \lesssim{}& \|u^{\varepsilon_1}\|_{L_t^2 H_x^1(|x|>R)} + \|u^{\varepsilon_2}\|_{L_t^2 H_x^1(|x|>R)} \\ &+ \|u^{\varepsilon_1} - u^{\varepsilon_2}\|_{L_t^2 H_x^1(|x|\le R)} \end{aligned}$$

(with all time integrals over $[0,T]$). The contribution of the terms on the region $|x| > R$ can be made small by choosing R large (uniformly in ε), while the contribution of the $|x| \le R$ tends to zero as $\varepsilon_1, \varepsilon_2 \to 0$ by local H^1 convergence. It follows that $\{u^\varepsilon\}$ is Cauchy in $L_t^2 H_x^1$ and hence we obtain convergence in this norm.

To complete the proof, we wish to show that u solves (8.1) in a weak sense. To this end, we take $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^d)$ and wish to prove

$$\int_0^T \int u\varphi_t\,dx\,dt = \int_0^T \int (u\varphi_{xxx} - 3u^2\varphi_x)\,dx\,dt.$$

In fact, we have

$$0 = \int_0^T \int [u_t^\varepsilon + u_{xxx}^\varepsilon + \varepsilon u_{xxxx}^\varepsilon - 3\partial_x[(u^\varepsilon)^2]\varphi\,dx\,dt \quad \text{for all} \quad \varepsilon > 0.$$

Now,

$$\iint u_t^\varepsilon \varphi\, dx\, dt = -\iint u^\varepsilon \varphi_t\, dx\, dt \to -\iint u\varphi_t\, dx\, dt \quad \text{as} \quad \varepsilon \to 0$$

by $L_t^2 L_x^2$ convergence. Similarly,

$$\iint u_{xxx}^\varepsilon \varphi\, dx\, dt = -\int u^\varepsilon \varphi_{xxx}\, dx\, dt \to -\int u\varphi_{xxx}.$$

Next,

$$\varepsilon \iint u_{xxxx}^\varepsilon \varphi\, dx\, dt = \varepsilon \iint u_{xx}^\varepsilon \varphi_{xx}\, dx\, dt = \mathcal{O}(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Finally, using $L_t^2 H_x^1$ convergence,

$$\iint -3\partial_x[(u^\varepsilon)^2]\varphi = \iint 3(u^\varepsilon)^2 \varphi_x \to \iint 3u^2 \varphi_x\, dx\, dt.$$

The result follows! □

8.1. **Special solutions.** We next compute some explicit solutions to KdV. Our presentation largely follows the excellent text [1].

Recall that the KdV equation takes the form

$$u_t + u_{xxx} - 6uu_x = 0.$$

We will look for *traveling wave* solutions of the form

$$u(t,x) = f(\xi), \quad \xi = x - ct, \quad c \text{ constant}.$$

In this case, the KdV equation reduces to

$$-cf' - 6ff' + f''' = 0.$$

Noting that $ff' = \frac{1}{2}(f^2)'$, we can integrate this equation to obtain

$$-cf - 3f^2 + f'' = A$$

where A is an arbitrary constant. We now multiply the equation by f' and integrate again to obtain

$$\tfrac{1}{2}(f')^2 = f^3 + \tfrac{1}{2}cf^2 + Af + B,$$

where B is another arbitrary constant.

Example 24 (Soliton)**.** *Suppose we impose the boundary conditions $f, f', f'' \to 0$ as $\xi \to \pm\infty$. In this case, we must have $A = B = 0$ and the equation reduces to*

$$(f')^2 = f^2(2f + c).$$

A real solution exists only if $2f + c \geq 0$ for all ξ. Now, the equation above is separable and can be integrated as follows:

$$\int \frac{df}{f(2f+c)^{\frac{1}{2}}} = \pm \int d\xi.$$

We now make the substitution $f = -\frac{1}{2}c \operatorname{sech}^2\theta$ (with $c \geq 0$). Then (using $\operatorname{sech}'\theta = -\operatorname{sech}\theta \tanh\theta$ and $1 - \operatorname{sech}^2\theta = \tanh^2\theta$), we have

$$df = c \operatorname{sech}^2\theta \tanh\theta\, d\theta \quad \textit{and} \quad 2f + c = c\tanh^2\theta.$$

Thus we obtain

$$\int \frac{df}{f(2f+c)^{\frac{1}{2}}} = -\tfrac{2}{\sqrt{c}} \int d\theta = \pm \int d\xi,$$

so that

$$\theta = \pm\tfrac{1}{2}\sqrt{c}\xi + C$$

for some C. We therefore obtain

$$f(x-ct) = -\tfrac{1}{2}c\,\mathrm{sech}^2\{\tfrac{1}{2}\sqrt{c}(x-ct-x_0)\},$$

where $x_0 \in \mathbb{R}$ is arbitrary. This is the solitary wave solution for KdV, and the parameter c acts as the speed as well as a scaling parameter.

By a coordinate change, we can reverse the sign of the nonlinearity and obtain the same solution with a $+$ sign instead of a $-$ sign. This sech^2 profile then corresponds to the famous wave of translation observed by John Scott Russell (and then derived mathematically by Boussinesq, as well as Korteveg and de Vries). Of course, KdV is only an approximation to the full water waves system; the sech^2 profile is then also an approximation to a solitary wave solution to the water waves equation in appropriate regimes.

Let us return to the general equation derived above. We seek real-valued, bounded solutions to

$$\tfrac{1}{2}(f')^2 = F(f) := f^3 + \tfrac{1}{2}cf^2 + Af + B.$$

Thus we are led to consider the degree 3 polynomial F, and in particular its zeros and the regions where it is nonnegative:

First suppose that F has a simple zero at $f = f_1$ (with $f(\xi_1) = f_1$, say). Then near $f = f_1$,

$$\tfrac{1}{2}(f'^2) = (f - f_1)F'(f_1) + \mathcal{O}((f-f_1)^2), \quad F'(f_1) \neq 0.$$

By considering a Taylor series expansion of f around $\xi = \xi_1$ (say), we claim that we can derive

$$f(\xi) = f_1 + \tfrac{1}{2}(\xi-\xi_1)^2 F'(f_1) + \mathcal{O}((\xi-\xi_1)^4) \tag{8.3}$$

near $\xi = \xi_1$. Thus we can conclude f has a local minimum or maximum at $\xi = \xi_1$.

Proof of (8.3). Expand

$$f(\xi) - f_1 = a_1(\xi-\xi_1) + a_2(\xi-\xi_1)^2 + a_3(\xi-\xi_1)^3 + \mathcal{O}(\xi-\xi_1)^4.$$

Then

$$\tfrac{1}{2}(f')^2 = [a_1(\xi-\xi_1) + \mathcal{O}(\xi-\xi_1)^2]F'(f_1) + \mathcal{O}((f-f_1)^2),$$

and since the left-hand side is $\frac{1}{2}a_1^2 + \mathcal{O}(\xi-\xi_1)$, we firstly derive $a_1 = 0$. This further shows that $\mathcal{O}((f-f_1)^2) = \mathcal{O}(\xi-\xi_1)^4)$.

Now return to the formula $\frac{1}{2}(f')^2 = F(f)$, which says

$$\tfrac{1}{2}[2a_2(\xi-\xi_1) + \mathcal{O}(\xi-\xi_1)^2]^2 = [a_2(\xi-\xi_1)^2 + \mathcal{O}(\xi-\xi_1)^3]F'(f_1) + \mathcal{O}((\xi-\xi_1)^4).$$

Matching the $(\xi-\xi_1)^2$ coefficients now yields $a_2 = \frac{1}{2}F'(f_1)$. If we can show $a_3 = 0$ then the result follows. We use the equation once more:

$$\begin{aligned}\tfrac{1}{2}&[2a_2(\xi-\xi_1) + 3a_3(\xi-\xi_1)^2 + \mathcal{O}(\xi-\xi_1)^3]^2 \\ &= [a_2(\xi-\xi_1)^2 + a_3(\xi-\xi_1)^3]F'(f_1) + \mathcal{O}((\xi-\xi_1)^4).\end{aligned}$$

Matching the $(\xi-\xi_1)^3$ terms and inserting $a_2 = \frac{1}{2}F'(f_1) \neq 0$ now implies

$$6a_2a_3 = a_3F'(f_1) \implies 3a_3 = a_3 \implies a_3 = 0,$$

as desired. □

Next, suppose that F has a double zero at $f = f_1$ (again with $f(\xi_1) = f_1$). In this case we have

$$(f')^2 = (f - f_1)^2 F''(f_1) + \mathcal{O}((f - f_1)^3),$$

and (considering the shape of F) this is only possible if $F''(f_1) > 0$. In this case, assuming a bounded solution, one obtains the asymptotics

$$f - f_1 \approx \alpha \exp\{-[F''(f_1)]^{\frac{1}{2}} |\xi|\} \quad \text{as} \quad |\xi| \to \infty.$$

Thus we obtain a solution with one peak defined for all $\xi \in \mathbb{R}$.

If F has a triple zero at $f = f_1$, the situation is somewhat degenerate and one can work out that $f(\xi) = -\frac{1}{6}c + 2(\xi - \beta)^{-2}$ for some β. We reject this as an unbounded solution.

So we can restrict to one of two scenarios: either

(i) f' changes sign around $f = f_1$, or

(ii) $f' \to 0$ as $|\xi| \to \infty$.

Now draw the possible shapes of F (see [1, page 23]).

Consider the case of a simple zero at f_1 and no other real zeros. If $f'(\xi_0) > 0$ for some ξ_0, then $F(\xi) > 0$ for all $\xi > \xi_0$ and we obtain that f is unbounded. If Instead $f'(\xi_0) < 0$ then f will decrease until it reaches $f = f_1$, at which point by (i) we see that f' changes sign. But then again we obtain unboundedness. So this case is not possible. We can similarly rule out the case of a double zero at some $f_3 < f_1$, as well as the case that f_1 is the largest of three simple zeros and $f > f_1$.

What other cases are possible? One possibility is a simple zero at f_3 and a double zero at $f_1 > f_3$. If $f \in (f_3, f_1)$ then we can derive that the solution attains a minimum at $f = f_3$ and satisfies $f \to f_1$ as $|\xi| \to \infty$. This is the solitary-wave solution with amplitude $f_3 - f_1 < 0$. (In this setting a solution with $f > f_1$ at any point must be unbounded.)

The remaining scenario is the situation in which there are three simple zeros $f_3 < f_2 < f_1$ and we have $f_3 < f < f_2$. In this case, f' changes sign at both f_3 and f_2. As the behavior of f near these points is algebraic (in particular, not exponential), the points at which $f = f_2$ and $f = f_3$ will be a finite distance apart. Any particular solution is determined completely if f and $\operatorname{sign}(f')$ are given at some point, which fixes a value between f_2 and f_3. The solution thereafter oscillates between f_2 and f_3 with a finite period. In fact, we claim that the period is given by

$$T = 2 \int_{f_3}^{f_2} \frac{df}{\{2F(f)\}^{\frac{1}{2}}},$$

so that the period depends only on the parameters A, B, c. To see this, first suppose $\xi_2 < \xi_3$ are such that $f(\xi_2) = f_2$ and $f(\xi_3) = f_3$, so that

$$T = 2(\xi_3 - \xi_2).$$

On the interval (ξ_2, ξ_3), the map $\xi \mapsto f(\xi)$ is a bijection. Denote the inverse map by $f \mapsto g(f)$. Then by the fundamental theorem of calculus and implicit differeniation of $g(f(\xi)) = \xi$,

$$T = 2[g(f_3) - g(f_2)] = 2\int_{f_3}^{f_2} g'(f)\,df$$
$$= 2\int_{f_3}^{f_2} \frac{df}{f'(g(f))} = 2\int_{f_3}^{f_2} \frac{df}{\{2F(f)\}^{\frac{1}{2}}},$$

as claimed.

The same considerations show that

$$\xi = \xi_3 \pm \int_{f_3}^{f(\xi)} \frac{dy}{\{2F(y)\}^{\frac{1}{2}}},$$

which provides an *implicit* formula for the solution f. Here $\pm$ is determined by whether $f'(\xi)$ is positive or negative.

8.2. **Elliptic functions.** In this section we will show that the implicit formula above can be written in terms of Jacobi elliptic functions and derive an expression for special solutions to KdV.

Let us introduce the following integral

$$v = \int_0^{\phi} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac{1}{2}}}, \quad 0 \le m \le 1. \tag{8.4}$$

We can use this formula relating v, ϕ to introduce the *Jacobi elliptic functions* $\mathrm{sn}(v|m)$ and $\mathrm{cn}(v|m)$ via

$$\mathrm{sn}(v|m) = \sin\phi, \quad \mathrm{cn}(v|m) = \cos\phi.$$

Often one writes $m = k^2$. Some basic terminology: in this case we may call m the 'parameter' and k the 'modulus'.

The cases $m = 0, 1$ are special cases because the integral (8.4) can be evaluated:

- If $m = 0$, then $v = \phi$, so that
$$\mathrm{cn}(v|0) = \cos(\phi) = \cos(v).$$
Similarly $\mathrm{sn}(v|0) = \sin(v)$.
- If $m = 1$, then $v = \mathrm{arcsech}(\cos\phi)$, so that
$$\mathrm{cn}(v|1) = \mathrm{sech}(v).$$
Similarly $\mathrm{sn}(v|1) = \tanh(v)$.

In fact, we now claim that $\mathrm{cn}(v|m)$ and $\mathrm{sn}(v|m)$ are periodic functions for all $m \in [0, 1)$ (while we see that periodicity fails at $m = 1$). To see this, let us introduce the *complete elliptic integral of the first kind*:

$$K(m) := \int_0^{\pi/2} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac{1}{2}}},$$

so that

$$4K(m) = \int_0^{2\pi} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac{1}{2}}}.$$

We claim that for $(0 \le m < 1)$ and any $\ell \in \mathbb{Z}$, we have

$$\mathrm{sn}(v + 4\ell K(m)|m) = \mathrm{sn}(v|m) \quad \text{and} \quad \mathrm{cn}(v + 4\ell K(m)|m) = \mathrm{cn}(v|m),$$

so that $\mathrm{sn}(\cdot|m)$ and $\mathrm{cn}(\cdot|m)$ are periodic with period $4K(m)$. Indeed, relating v, ϕ as in (8.4), we have

$$\begin{aligned} v + 4K(m) &= \int_0^{\phi} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac12}} + \int_0^{2\pi} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac12}} \\ &= \int_{2\pi}^{\phi+2\pi} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac12}} + \int_0^{2\pi} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac12}} \\ &= \int_0^{\phi+2\pi} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac12}}. \end{aligned}$$

Thus if $\mathrm{sn}(v|m) = \sin(\phi)$, then

$$\mathrm{sn}(v + 4K(m)|m) = \sin(\phi + 2\pi) = \sin(\phi) = \mathrm{sn}(v|m).$$

The claimed periodicity follows.

This calculation breaks down as $m \to 1$ since $K(m) \to \infty$ as $m \to 1$. However, for any $0 \le m < 1$, we obtain periodicity with a finite period. In particular $K(0) = \frac{\pi}{2}$, and one can show that $K(m) \sim \frac12 \log(\frac{16}{1-m})$ as $m \to 1$.

Using the fact that $\cos^2\phi + \sin^2\phi = 1$, we immediately obtain the identity

$$\mathrm{cn}^2 + \mathrm{sn}^2 = 1.$$

We can also derive identities relating derivatives of elliptic functions. Begin again with (8.4):

$$v = \int_0^{\phi} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac12}}, \quad \phi = \phi(v).$$

Differentiate with respect to v to obtain

$$1 = \frac{\partial_v \phi}{(1 - m\sin^2(\phi(v)))^{\frac12}} \implies \partial_v\phi = (1 - m\sin^2\phi(v))^{\frac12}.$$

In fact, with v, ϕ related by (8.4), we can introduce an additional elliptic function $\mathrm{dn}(v|m)$ via

$$\mathrm{dn}(v|m) = (1 - m\sin^2\phi)^{\frac12}$$

and thereby obtain

$$\begin{aligned} \partial_v \,\mathrm{cn}(v|m) &= -\sin(\phi)\partial_v\phi \\ &= -\,\mathrm{sn}(v|m)\,\mathrm{dn}(v|m). \end{aligned}$$

Similarly $\partial_v \,\mathrm{sn}(v|m) = \mathrm{cn}(v|m)\,\mathrm{dn}(v|m)$, and we can observe the identity

$$\mathrm{dn}^2(v|m) + m\,\mathrm{sn}^2(v|m) = 1.$$

With these preliminaries in place, let us return to the periodic solution $f(\xi)$ derived in the previous section, which satisfies:

$$\tfrac12 (f')^2 = F(f) := f^3 + \tfrac12 cf^2 + Af + B = (f - f_1)(f - f_2)(f - f_3),$$

with $f_3 < f_2 < f_1$. We derived the following implicit formula for f:

$$\xi = \xi_3 \pm \int_{f_3}^{f(\xi)} \frac{dg}{\{2(g - f_1)(g - f_2)(g - f_3)\}^{\frac12}},$$

where $f(\xi_3) = f_3$. Now let us make the change of variables

$$g = f_3 + (f_2 - f_3)\sin^2\theta, \quad \text{so that} \quad dg = 2(f_2 - f_3)\sin\theta\cos\theta\, d\theta.$$

Being careful with minus signs and using $1-\sin^2\theta=\cos^2\theta$, we can simplify the function in the denominator as

$$([f_1-f_3]-(f_2-f_3)\sin^2\theta)^{\frac12}(f_2-f_3)^{\frac12}\cos\theta(f_2-f_3)^{\frac12}\sin\theta.$$

Thus we obtain

$$\frac{dg}{\{2(g-f_1)(g-f_2)(g-f_3)\}^{\frac12}}=\frac{\sqrt{2}}{(f_1-f_3)^{\frac12}}\frac{d\theta}{(1-\frac{f_2-f_3}{f_1-f_3}\sin^2\theta)^{\frac12}}.$$

We further introduce ϕ and m so that

$$f(\xi)=f_3+(f_2-f_3)\sin^2\phi \quad \text{and} \quad m=\frac{f_2-f_3}{f_1-f_3}.$$

Then we obtain

$$\xi=\xi_3\pm\big(\tfrac{2}{f_1-f_3}\big)^{\frac12}\int_0^\phi\frac{d\theta}{(1-m\sin^2\theta)^{\frac12}},$$

so that

$$\pm\int_0^\phi\frac{d\theta}{(1-m\sin^2\theta)^{\frac12}}=\big(\tfrac{f_1-f_3}{2}\big)^{\frac12}[\xi-\xi_3].$$

Rewriting

$$f(\xi)=f_2-(f_2-f_3)\cos^2\phi,$$

we derive

$$f(\xi)=f_2-(f_2-f_3)\,\mathrm{cn}^2((\tfrac{f_1-f_3}{2})^{\frac12}[\xi-\xi_3]|\tfrac{f_2-f_3}{f_1-f_3}),$$

where $f(\xi_3)=f_3$. Amazing!

Given a choice of parameters $f_3<f_2<f_1$, let us discuss the shape of the function $f(\xi)$ obtained above. (Recall that we might wish to change coordinates in such a way that f is replaced with $-f$; here we stick with the coordinates we have been using.)

The parameter f_2 determines the maximum value of f, while f_3 is its minimum value in one period. We call $\frac12(f_2-f_3)$ the 'amplitude' of the wave. The function is periodic and its wavelength or period is given by

$$2K(\tfrac{f_2-f_3}{f_1-f_3})\cdot\big(\tfrac{2}{f_1-f_3}\big)^{\frac12}$$

(note that $\mathrm{cn}^2(v|m)$ has period $2K(m)$, rather than $4K(m)$).

Now recall that the original derivation of the traveling wave solution supposed that we have a solution of the form $u(t,x)=f(x-ct)$. What is the parameter c in terms of f_1,f_2,f_3? To determine this, let us recall that we are assuming

$$\tfrac12(f')^2=F(f)=f^3+\tfrac12cf^2+Af+B$$

and that

$$F=(f-f_1)(f-f_2)(f-f_3).$$

Matching powers, we find

$$c=-2(f_1+f_2+f_3).$$

Thus we obtain a solution to KdV of the form

$$u(t,x)=f_2-(f_2-f_3)\,\mathrm{cn}^2((\tfrac{f_1-f_3}{2})^{\frac12}(x+2(f_1+f_2+f_3)t)|\tfrac{f_2-f_3}{f_1-f_3}).$$

Note that *the speed, shape, and wavelength all depend on the amplitude of the wave in a complicated manner.*

In the case that $m = \frac{f_2 - f_3}{f_1 - f_3} \to 0$ we expect to recover the 'linear' limit, while in the case $m = \frac{f_2 - f_3}{f_1 - f_3} \to 1$ we should recover the solitary wave solution.

9. Dispersive shocks

References: Whitham Section 16.14, paper of Gurevich–Pitaevskii

In this section we wish to discuss *dispersive shock wave* solutions for KdV. In the previous section we wrote KdV in the form

$$v_t - 6vv_x + v_{xxx} = 0.$$

The basic premise is that we expect the dispersive term u_{xxx} to have a 'regularizing' effect on the shock solutions we previous constructed for the pure Burgers-type equation. There is a problem with our current formulation, though. Note that with the sign convention above, the Burgers shock (for the equation $u_t - 6uu_x = 0$) would form for data with $u_L < u_R$ and move to the *left*, while the KdV solitons (which will play a role in the analysis) move to the *right*. Fortunately, this is all easy to remedy: Note that if $u(t, x) = -v(t, x)$, then v solves KdV if and only if u solves

$$u_t + 6uu_x + u_{xxx} = 0.$$

In this case, the underlying Burgers equation $u_t + 6uu_x$ will form shocks when $u_L > u_R$ and the shocks will move to the right. Thus this is the right model to use to introduce dispersive shock waves. Moreover, all of our hard work finding special solutions in the previous section is not wasted: we simply need to multiply the special solutions we found by -1.

We turn to the details: Recall that for the Burgers type equation

$$u_t + 6uu_x = 0$$

with step-like data

$$u|_{t=0} = \begin{cases} 1 & x < 0 \\ 0 & x > 0, \end{cases}$$

we obtain an instantaneous shock starting from $(t, x) = (0, 0)$, namely:

$$u(t, x) = \begin{cases} 1 & x < 3t \\ 0 & x > 3t. \end{cases}$$

We wish to construct an (approximate) solution to KdV with such initial data and to understand its structure as $t \to \infty$.

Recalling the change in sign convention discussed above, we proved in the previous section that for any $f_3 < f_2 < f_1$, we have the following special solution to the KdV ($u_t + 6uu_x + u_{xxx} = 0$):

$$u(t, x) = f(x - ct), \quad c := -2(f_1 + f_2 + f_3),$$

where

$$f(\xi) = -f_2 + (f_2 - f_3)\,\mathrm{cn}^2\big((\tfrac{f_1 - f_3}{2})^{\frac{1}{2}}\xi \,|\, \tfrac{f_2 - f_3}{f_1 - f_3}\big)$$

The function f is a periodic wave taking values in $[-f_2, -f_3]$ with

- 'amplitude' a equal to $\frac{1}{2}(f_2 - f_3)$, and
- period equal to $2K(\frac{f_2 - f_3}{f_1 - f_3})(\frac{2}{f_1 - f_3})^{\frac{1}{2}}$.

The idea now is that to model a shock-type solution with step-like data, we will allow the parameters f_1, f_2, f_3 to vary as functions of (t,x). In the 'shock region' we will have $f_3 < f_2 < f_1$, while on the left/right of the shock the situation will degenerate to $f_3 = f_2 < f_1$ or $f_3 < f_2 = f_1$, respectively.

More precisely, we assume that the parameters depend only on the ratio $\tau := \frac{x}{t}$, i.e. $f_j = f_j(\tau)$. By considering the behavior of the Burgers shock solution, we expect the solution to satisfy $u \equiv 1$ to the left of some value, say $\tau = \tau^-$, and $u \equiv 0$ to the right of some value, say $\tau = \tau^+$. At this moment the values $\tau^\pm$ have not been determined.

We will then set

$$f_2(\tau) \equiv f_3 = -1 \quad \text{for} \quad \tau \leq \tau^-,$$

while

$$f_2(\tau) \equiv f_1 = 0 \quad \text{for} \quad \tau \geq \tau^+.$$

Note that in the case $\tau \geq \tau^+$, the parameter $m = \frac{f_2 - f_3}{f_1 - f_3} \equiv 1$ (giving the soliton solution).

For $\tau^- \leq \tau \leq \tau^+$, we will have $-1 \leq f_2(\tau) \leq 0$ and $0 \leq m \leq 1$.

So far, we can write our ansatz for the dispersive shock solution in the form

$$u(t,x) = -f_2(\tau) + [f_2(\tau)+1]\,\mathrm{cn}^2\big(\tfrac{1}{\sqrt{2}}(x + 2(f_2(\tau)-1)t)|1 + f_2(\tau)\big), \quad \tau = \tfrac{x}{t}.$$

In particular, recalling that $\mathrm{cn}(v|1) = \mathrm{sech}(v)$, we have

$$u(t,x) = \begin{cases} 1 & \tau \leq \tau^- \\ \mathrm{sech}^2\big(\tfrac{1}{\sqrt{2}}(x-2t)\big) & \tau \geq \tau^+ \end{cases}$$

To complete our ansatz, we need to specify the values $\tau^\pm$ and propose a suitable evolution for the parameter $f_2(\tau)$. We can equivalently parametrize the solution via $m(\tau) = 1 + f_2(\tau)$, in which case our solution has the form

$$u(t,x) = 1 - m + m\,\mathrm{cn}^2\big(\tfrac{1}{\sqrt{2}}(x - 4t + 2mt)|m\big), \quad m = m(\tau).$$

We know from above that $m \equiv$ const yields an exact solution to KdV. Thus we expect that a slowly varying m will yield an approximate solution.

To make the notation more compact, let us denote

$$u = A + B\Phi(v|m),$$

where $A = 1-m$, $B = m$, $\Phi = \mathrm{cn}^2$, and $v = \frac{1}{\sqrt{2}}[x - (4-2m)t]$. We use subscripts to denote derivatives.

It is possible, but rather complicated, to explicitly compute $u_t + u_{xxx} + 6uu_x$ for u of the form above. So we will instead introduce some approximations.

First, recall that we are assuming assuming $m = m(\tau)$ with $\tau = \frac{x}{t}$, so that for example we can write

$$m_x = \tfrac{1}{t}m' \quad \text{and} \quad m_t = -\tfrac{x}{t^2}m',$$

where $m' = \partial_\tau m$. Derivatives of m will be supported on $\tau \in [\tau^-, \tau^+]$ and we have $\int_{\tau^-}^{\tau^+} m'(\tau)\,d\tau = 1$. In particular we have that $|\frac{x}{t}|$ is bounded on the support of m' and we can take m' itself to be bounded. Thus, roughly speaking, every differentiation of m (with respect to t or x) produces a factor of t^{-1}.

We are interested in producing an asymptotically correct solution to KdV (as $t \to \infty$). Thus, in what follows, we will regard any term that is $\mathcal{O}(t^{-1})$ (or smaller) as a perturbation and ultimately disregard it completely. Note that the derivative of a term that has been treated as a perturbation can still safely be treated as a perturbation; indeed, differentiation only ever introduces additional $\mathcal{O}(1)$ or $\mathcal{O}(t^{-1})$ terms.

So, for example, with $u = A + B\Phi$ we have

$$\begin{aligned} u_t &= A_t + B_t\Phi + B\Phi_v v_t + B\Phi_m m_t \\ &= \tfrac{1}{\sqrt{2}}(-4 + 2m + 2tm_t)B\Phi_v + \mathcal{O}(t^{-1}). \end{aligned}$$

Similarly,

$$\begin{aligned} u_x &= A_x + B_x\Phi + B\Phi_v v_x + B\Phi_m m_x \\ &= \tfrac{1}{\sqrt{2}}(1 + 2tm_x)B\Phi_v + \mathcal{O}(t^{-1}). \end{aligned}$$

(A word about notation: Φ and its derivatives are all evaluated at $(v|m)$; the parentheses indicate a pointwise product.) Further computation shows

$$u_{xx} = [\tfrac{1}{\sqrt{2}}(1 + 2tm_x)]^2 B\Phi_{vv} + \mathcal{O}(t^{-1}),$$

and similarly

$$u_{xxx} = [\tfrac{1}{\sqrt{2}}(1 + 2tm_x)]^3 B\Phi_{vvv} + \mathcal{O}(t^{-1}).$$

Thus

$$\begin{aligned} &u_t + u_{xxx} + 6uu_x \\ &\quad = \tfrac{1}{\sqrt{2}}(-4 + 2m + 2tm_t)B\Phi_v + [\tfrac{1}{\sqrt{2}}(1 + 2tm_x)]^3 B\Phi_{vvv} \\ &\qquad\quad + 6(A + B\Phi)\tfrac{1}{\sqrt{2}}(1 + 2tm_x)B\Phi_v + \mathcal{O}(t^{-1}). \end{aligned}$$

We now recall that if m is constant, then u is an *exact* solution to KdV. Thus the computations above (setting $m_t = m_x = 0$) yield

$$\tfrac{1}{\sqrt{2}}(-4 + 2m)B\Phi_v + [\tfrac{1}{\sqrt{2}}]^3 B\Phi_{vvv} + 6(A + B\Phi)\tfrac{1}{\sqrt{2}}B\Phi_v = 0.$$

Inserting this identity in to the preceding display, we obtain

$$\begin{aligned} &u_t + u_{xxx} + 6uu_x \\ &\quad = \tfrac{1}{\sqrt{2}}2tm_t B\Phi_v + (\tfrac{1}{\sqrt{2}})^3[6tm_x + 12(tm_x)^2 + 8(tm_x)^3]B\Phi_{vvv} \\ &\qquad\quad + 12(A + B\Phi)\tfrac{1}{\sqrt{2}}tm_x B\Phi_v + \mathcal{O}(t^{-1}). \end{aligned}$$

Writing

$$tm_t = -\tfrac{x}{t}m' \quad \text{and} \quad tm_x = m'$$

and simplifying a bit, the equation can be further reduced to

$$\begin{aligned} &\sqrt{2}[u_t + u_{xxx} + 6uu_x] \\ &\qquad = -2\tfrac{x}{t}m'B\Phi_v + \{3m' + 6(m')^2 + 4(m')^3\}B\Phi_{vvv} \\ &\qquad\quad + 12(A + B\Phi)m'B\Phi_v + \mathcal{O}(t^{-1}) \end{aligned}$$

We make our first approximation by taking the right-hand side to be identically zero. We then divide the entire equation by $2m'B\Phi_v$, recall the definition of A, B, and simplify. This yields

$$-\tfrac{x}{t} + \tfrac{\Phi_{vvv}}{\Phi_v}\{\tfrac{3}{2} + 3m' + 2(m')^2\} + 6(1 - m + m\Phi) = 0.$$

Now we use the following:

Lemma 9.1. *The function Φ satisfies*

$$\tfrac{\Phi_{vvv}}{\Phi_v} = -4 + 8m - 12m\Phi.$$

Proof. This can be proved using various formulas relating derivatives of the elliptic functions. We can also argue a bit more simply as follows: Fix m and let $y(v) = \mathrm{cn}(v|m)$. It is well-known that y satisfies the ODE

$$y'' + (1-2m)y + 2my^3 = 0.$$

In fact this ODE can be derived by manipulating the types of formulas we saw in the previous section. Since $\Phi = y^2$, we have

$$\Phi_v = 2yy' \quad \text{and} \quad \Phi_{vv} = 2yy'' + 2(y')^2.$$

We now insert the equation for y'' to get

$$\Phi_{vv} = -2(1-2m)y^2 - 4my^4 + 2(y')^2.$$

Now differentiate one more time to get

$$\begin{aligned} \Phi_{vvv} &= -4(1-2m)yy' - 16my^3y' + 4y'y'' \\ &= -2(1-2m)\Phi_v - 8m\Phi\Phi_v + 2\Phi_v \tfrac{y''}{y}. \end{aligned}$$

Then, using

$$\tfrac{y''}{y} = -(1-2m) - 2my^2 = -(1-2m) - 2m\Phi,$$

we have

$$\frac{\Phi_{vvv}}{\Phi_v} = -2(1-2m) - 8m\Phi - 2(1-2m) - 4m\Phi$$

Simplifying now yields the result. □

Inserting the result of the lemma, the equation above becomes

$$-\tfrac{x}{t} + (-4 + 8m - 12m\Phi)(\tfrac{3}{2} + 3m' + 2(m')^2) + 6(1 - m + m\Phi)) = 0.$$

The implicit dependence of Φ on m makes things rather confusing, so now we make our next move. We are going to 'average over one period of Φ'. Recall that $\Phi = \Phi(v|m)$, with $v = \frac{1}{\sqrt{2}}(x - 2mt)$ and $m = m(\tau)$, $\tau = \frac{x}{t}$. We take the mindset that $m(\tau)$ is slowly varying, while v is the 'fast variable'. So we can integrate over one period with respect to v while essentially treating $m(\tau)$ as constant. One can compute

$$\frac{1}{2K(m)} \int_0^{2K(m)} \mathrm{cn}^2(v|m)\,dv = \frac{1}{m}\left[\frac{E(m)}{K(m)} - (1-m)\right],$$

where

$$E(m) = \int_0^{\frac{\pi}{2}} (1 - m\sin^2\theta)^{\frac{1}{2}}\,d\theta \quad \text{and} \quad K(m) = \int_0^{\frac{\pi}{2}} (1 - m\sin^2\theta)^{-\frac{1}{2}}\,d\theta.$$

(Exercise: prove this!) After some algebraic simplification, the result is then the following:

$$\tau = (8 - 4m - 12\tfrac{E(m)}{K(m)})(\tfrac{3}{2} + 3m' + 2(m')^2) + 6(\tfrac{E(m)}{K(m)}).$$

We are getting closer now. We can view this as a first order (nonlinear) ODE for m. Or, we could throw away the $(m')^2$ term and obtain at least a simpler ODE for m. Or, if we take the view that if m is really slowly varying, we could throw away all the derivatives in our approximation and just obtain an *algebraic* equation for m! That sounds easiest (although the least accurate), so let's go for that. Some algebraic manipulation yields

$$\tau = 6(2 - m - 2\tfrac{E(m)}{K(m)}). \tag{9.1}$$

This is an implicit formula that we can use to define $m = m(\tau)$. To determine $\tau^{\pm}$, we need to use the conditions $m \to 0$ as $\tau \to \tau^-$ and $m \to 1$ as $\tau \to \tau^+$. We have that

$$\lim_{m\to 0} \tfrac{E(m)}{K(m)} = 1 \quad \text{and} \quad \lim_{m\to 1} \tfrac{E(m)}{K(m)} = 0.$$

From this we can take limits as $m \to 0$ and $m \to 1$ in the formula for τ to obtain

$$\tau^- = 0 \quad \text{and} \quad \tau^+ = 6.$$

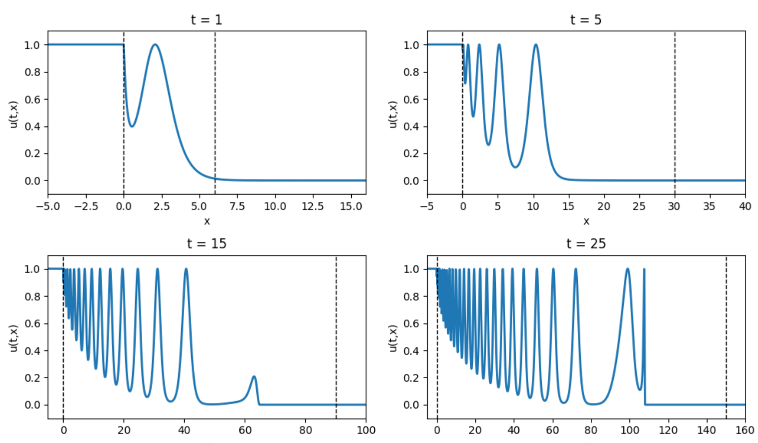
Thus we can complete our ansatz and finally obtain:

$$u(t,x) = \begin{cases} 1 & x \leq 0 \\ 1 - m + m\,\mathrm{cn}^2(\tfrac{1}{\sqrt{2}}(x - 4t + 2mt)|m) & 0 \leq x \leq 6t \\ \mathrm{sech}^2(\tfrac{1}{\sqrt{2}}(x - 2t)) & x \geq 6t, \end{cases}$$

where $m = m(\frac{x}{t}) \in [0,1]$ is defined implicitly for $0 \leq x \leq 6t$ via

$$\tfrac{x}{t} = \tau = 6(2 - m - 2\tfrac{E(m)}{K(m)}).$$

Below is a snapshot of the solution at several different times (the code to produce the plot is courtesy of ChatGPT, so who knows how accurate it is).



We have presented one simple example and derived the algebraic condition for m by using the KdV equation directly. Let us just mention that this is a very special case (with self-similar parameters) of a more general theory known as Whitham modulation theory (see [8, 9]).

10. Soliton stability

References: Tao [7]

Notation: Throughout this section, we will use $\langle \cdot, \cdot \rangle$ to denote the real L^2 inner product.

In this section we continue to study the KdV equation

$$u_t + u_{xxx} + 6uu_x = 0.$$

Recall that in the previous section, we obtained local-in-time weak solutions for H^2-data via energy methods. Here we will take for granted stronger existence results, e.g. the results of Kenig–Ponce–Vega yielding global well-posedness in H^1 (see e.g. [4]). In particular H^1 data leads to global H^1 solutions, with $t \mapsto \|u(t)\|_{H^1}$ continuous. These results rely on more sophisticated harmonic analysis tools and hence are beyond the scope of these notes.

We turn our attention to properties of the *soliton solution*

$$u_c(t,x) = \tfrac{1}{2}c\,\mathrm{sech}^2\{\tfrac{1}{2}\sqrt{c}(x-ct)\} =: Q_c(x-ct), \quad c \geq 0.$$

In particular, we are interested in *stability properties* of these special solutions. Roughly speaking, if we start with initial data near some soliton, will the solution remain near the corresponding soliton solution for all times $t \geq 0$?

Stated this way, the answer is *no.* Indeed, by choosing c_1 and c_2 sufficiently close, we can make Q_{c_1} and Q_{c_2} as close as we wish (in any reasonable norm). However, as soon as $c_1 \neq c_2$, the two solutions u_{c_1} and u_{c_2} will eventually become widely separated compared to their characteristic supports, and in particular will not be close in any reasonable norm.

Thus we need to modify our notion of stability. In particular, we will begin by introducing the notion of *orbital stability*:

Definition 4 (Orbital stability)**.** *We call the soliton Q* orbitally stable *(in H^1) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\|u_0 - Q\|_{H^1} < \delta,$$

then the solution u to KdV with $u|_{t=0} = u_0$ satisfies

$$\sup_{t\geq 0} \inf_{x_0 \in \mathbb{R}} \|u(t) - Q(\cdot - x_0)\|_{H^1} < \varepsilon.$$

This is at least compatible with the example provided above, by observing that $|c_2 - c_1| \ll 1$ implies

$$\|u_{c_2}(t,x) - Q_{c_1}(x - c_2 t)\|_{H^1} = \|Q_{c_2}(x-c_2t) - Q_{c_1}(x-c_2t)\|_{H^1} \ll 1$$

uniformly in t.

Based on the general sketch in [7][2], we will prove the following theorem.

Theorem 10.1. *The KdV soliton Q is orbitally stable in H^1.*

We use a version of the *Lyapunov function* method (as in the theory of ODE). We define

$$L(u) = M(u) + E(u),$$

where

$$M(u) = \int \tfrac{1}{2}u^2\,dx \quad \text{and} \quad E(u) = \int \tfrac{1}{2}u_x^2 - u^3\,dx.$$

The quantities M and E have an interpretation as the *mass* and *energy*, and one can verify that these quantities are conserved in time (at least formally), so that $L(u(t)) \equiv L(u(0))$. It is worth mentioning that $E(u)$ is finite for H^1 data. Indeed, we have $H^1 \subset L^2 \cap L^\infty \subset L^3$.

[2] I am also grateful to Renzo Scarpelli for discussions related to coercivity.

The functional L is continuous on H^1, in the sense that $u_n \to u$ in H^1 guarantees $L(u_n) \to L(u)$. Note also that L is translation-invariant. In particular, if u_0 is close to Q in H^1, then $|L(u_0)-L(Q)| \ll 1$. But then by conservation laws and translation-invariance, $|L(u(t))-L(\tau Q)| \ll 1$ for all $t \geq 0$ and for any translation τQ of Q. We will try to argue that there exist suitable choices of time-dependent translates of Q such that $|L(u(t)) - L(\tau Q)|$ essentially controls the difference $u(t) - \tau Q$ in H^1. This would yield orbital stability.

To get started, we are going to compute some *functional derivatives* of L in order to understand the 'shape' of L near Q.

We have that $L : H^1 \to \mathbb{R}$. Its derivative at $u \in H^1$ will be the linear map $L'(u) : H^1 \to \mathbb{R}$, defined by property that

$$\tfrac{1}{\varepsilon}\big[L(u+\varepsilon v) - L(u) - \varepsilon L'(u)v\big] \to 0 \quad \text{in} \quad H^1 \quad \text{as} \quad \varepsilon \to 0.$$

for any $v \in H^1\backslash\{0\}$. So, given $L : H^1 \to \mathbb{R}$, we have $L : H^1 \to \mathcal{L}(H^1;\mathbb{R})$.

To compute L', we should expand $L(u+\varepsilon v)$. Let's compute this for M and E separately:

$$M(u+\varepsilon v) = \int \tfrac{1}{2}u^2 + \varepsilon uv + \tfrac{1}{2}\varepsilon^2 v^2\,dx, \quad \text{so that} \quad M'(u)v = \int uv\,dx.$$

We may abbreviate this $M'(u) = u$. Next,

$$\begin{aligned} E(u+\varepsilon v) = &\int \tfrac{1}{2}u_x^2 + \varepsilon u_x v_x + \tfrac{1}{2}\varepsilon^2 v_x^2\,dx \\ &- \int [u^3 + 3\varepsilon u^2 v + 3\varepsilon^2 uv^2 + \varepsilon^3 v^3]\,dx. \end{aligned}$$

Now note that

$$\varepsilon \int u_x v_x\,dx = -\varepsilon \int u_{xx} v\,dx.$$

Thus

$$E'(u)v = \int -u_{xx}v - 3u^2 v\,dx, \quad \text{or} \quad E'(u) = -u_{xx} - 3u^2.$$

Putting the pieces together, we find

$$L'(u)v = \int [u - u_{xx} - 3u^2]v, \quad \text{or} \quad L'(u) = u - u_{xx} - 3u^2.$$

If you recall, this quantity is exactly related to the ODE we derived for the soliton! Indeed:

$$L'(Q) = 0.$$

Thus Q is a critical point of L. To understand the shape of L near Q, we need to compute a second derivative. Recall that $L : H^1 \to \mathbb{R}$ and $L' : H^1 \to \mathcal{L}(H^1;\mathbb{R})$. Thus we will have $L'' : H^1 \to \mathcal{L}(H^1;\mathcal{L}(H^1;\mathbb{R}))$. In particular, given $u \in H^1$ we will have that $L''(u) \in \mathcal{L}(H^1;\mathcal{L}(H^1;\mathbb{R}))$, so that $L''(u)v : H^1 \to \mathbb{R}$. The defining property is that for $u, v \in H^1$,

$$\tfrac{1}{\varepsilon}[L'(u+\varepsilon v) - L'(u) - \varepsilon L''(u)v] \to 0 \quad \text{in} \quad \mathcal{L}(H^1;\mathbb{R}) \quad \text{as} \quad \varepsilon \to 0.$$

So, take $\psi \in H^1(\mathbb{R})$ and let's expand the quantity $L'(u+\varepsilon v)\psi$:

$$L'(u+\varepsilon v)\psi = \int [u + \varepsilon v - (u+\varepsilon v)_{xx} - 3(u+\varepsilon v)^2]\psi\,dx.$$

Isolating the $\mathcal{O}(\varepsilon)$ terms, we derive

$$\begin{aligned} L''(u)v\psi &= \int v\psi - v_{xx}\psi - 6uv\psi\,dx \\ &= \int v\psi + v_x\psi_x - 6uv\psi\,dx. \end{aligned}$$

(We may view $L''(u)$ as a quadratic form and write $L''(u) = 1 - \partial_x^2 - 6u$.)

We have the following Taylor expansion:

$$L(u+v) = L(u) + L'(u)v + \tfrac{1}{2}L''(u)vv + \mathcal{O}(\|v\|_{H^1}^3).$$

(This can be derived by applying the $1d$ Taylor expansion to $\ell(t) = L(u+tv)$, which satisfies $\ell'(t) = L'(u+tv)v$.)

In the orbital stability problem, we define parameters $x(t) \in \mathbb{R}$ and $v(t) \in H^1$ via

$$u(t,x) = Q(x - x(t)) + v(t, x - x(t)),$$

although we don't yet know how to choose $x(t)$. To prove orbital stability requires that we can choose $x(t)$ so that v remains uniformly small in H^1.

No matter what choice of $x(t)$ we make, since L is translation invariant and $L'(Q) = 0$, we can derive

$$\begin{aligned} L(u) &= L(Q) + \tfrac{1}{2}L''(Q)vv + \mathcal{O}(\|v\|_{H^1}^3) \\ &= L(Q) + \tfrac{1}{2}\int v^2 + v_x^2 - 6Qv^2\,dx + \mathcal{O}(\|v\|_{H^1}^3). \end{aligned}$$

The question is now whether we can choose $x(t)$ so that this final integral controls $\|v\|_{H^1}^2$. This is the question of *coercivity* of the quadratic form

$$\langle Bv, v\rangle, \quad B := 1 - \partial_x^2 - 6Q.$$

The dream would be an estimate like

$$\langle Bv, v\rangle \gtrsim \|v\|_{H^1}^2 \quad \text{for all} \quad v \in H^1 \quad ?$$

Unfortunately there is no way an estimate like this can hold: using $Q - Q'' - 3Q^2 = 0$, we have $BQ = -3Q^2$, so that

$$\langle BQ, Q\rangle = -3\|Q\|_{L^3}^3 < 0.$$

So B has at least one negative direction. Moreover, using the equation once again, we have

$$BQ' = 0, \quad \text{so that} \quad \langle BQ', Q'\rangle = 0.$$

The hope is that if we somehow avoid these two bad directions, we can obtain coercivity. However, it seems that we only have one parameter to play with, namely, the translation parameter $x(t)$. In fact, there is one other symmetry that we can get involved, namely, that of *scaling symmetry*. This refers to the fact that the class of solutions to KdV is invariant under the map

$$u(t,x) \mapsto cu(c^{\frac{3}{2}}t, c^{\frac{1}{2}}x)$$

for any $c > 0$. What we will eventually do is prove orbital stability under the additional assumption $\|u_0\|_{L^2} = \|Q\|_{L^2}$ (giving a second condition that we can leverage to obtain coercivity), and then play with the scaling symmetry to recover the general orbital stability result.

At this point, hopefully it is clear that key issue is the coercivity of B. We will take some 'ODE facts' for granted throughout the arguments below. For example, we have

$$N(B) = \text{span}\{Q'\},$$

where $N(B)$ denotes the null space of B.

There are a few other 'characters' that will play a role in the analysis below. Explicit calculation shows that with

$$W = -\tfrac{1}{2}xQ' \quad \text{and} \quad Z := -\tfrac{1}{2}xQ' - Q,$$

we have

$$B(W) = Q'' \quad \text{and} \quad B(Z) = Q.$$

Moreover,

$$\langle Z, Q\rangle = -\tfrac{3}{4}\langle Q, Q\rangle \quad \text{and} \quad \langle Z, Q'\rangle = 0.$$

Let us first prove a general lemma that shows that coercivity in L^2-norm automatically implies coercivity in H^1:

Lemma 10.1. *Suppose X is a subspace of H^1 such that*

$$\langle Bv, v\rangle \gtrsim \|v\|_{L^2}^2 \quad \textit{for all} \quad v \in X.$$

Then

$$\langle Bv, v\rangle \gtrsim \|v\|_{H^1}^2 \quad \textit{for all} \quad v \in X.$$

Proof. We have

$$\langle Bv, v\rangle = \|v\|_{H^1}^2 - 6\langle Qv, v\rangle.$$

Suppose towards a contradiction that there exist $v_n \in X$ such that

$$\langle Bv_n, v_n\rangle \le \tfrac{1}{n}\|v_n\|_{H^1}^2.$$

Then we have

$$\langle Bv_n, v_n\rangle = \|v_n\|_{H^1}^2 - 6\langle Qv_n, v_n\rangle \le \tfrac{1}{n}\|v_n\|_{H^1}^2.$$

Thus (for n large) we deduce

$$\|v_n\|_{H^1}^2 \lesssim |\langle Qv_n, v_n\rangle| \lesssim \|v_n\|_{L^2}^2.$$

But then

$$\langle Bv_n, v_n\rangle \lesssim \tfrac{1}{n}\|v_n\|_{L^2}^2,$$

contradicting coercivity with respect to the L^2-norm. □

We already saw that Q is a negative direction for B. We now prove that as long as we avoid this direction, B is at least nonnegative.

Lemma 10.2. *We have*

$$\alpha := \inf\left\{\frac{\langle Bv, v\rangle}{\langle v, v\rangle} : v \in H^1\backslash\{0\} \quad \textit{such that} \quad \langle v, Q\rangle = 0\right\} = 0.$$

Proof. As $BQ' = 0$ and $\langle Q, Q'\rangle = 0$, we have $\alpha \le 0$. We suppose towards a contradiction that $\alpha < 0$. It is not hard to check that $\alpha \ge -C$ for some $C > 0$.

We first show that the infimum is attained. We take an optimizing sequence v_n, which (by normalizing) can be assumed to satisfy $\langle v_n, v_n\rangle \equiv 1$. Then since

$$\langle Bv_n, v_n\rangle = \|v_n'\|_{L^2}^2 + 1 - 6\int Qv_n^2\,dx \to \alpha,$$

we have that $\{v_n\}$ is bounded in H^1. We can therefore pass to a subsequence and obtain a weak limit $v_n \rightharpoonup v$ in H^1. Using the decay of Q and Rellich's Theorem, we can obtain $\int Qv_n^2 \to \int Qv^2$. It follows (from weak lowersemicontinuity) that

$$\|v'\|_{L^2}^2 + 1 - 6\int Qv^2\,dx \leq \alpha.$$

We have that $\|v\|_{L^2} \leq 1$. In fact, we claim $\|v\|_{L^2} = 1$. Indeed, if $\|v\|_{L^2} < 1$ then (since $\alpha < 0$) the inequality above would imply that v is a 'super-optimizer'. We therefore deduce that v attains the infimum α and satisfies the orthogonality condition $\langle v, Q\rangle = 0$. In light of the constrained minimization problem that v solves, we derive that

$$(B-\alpha)v = \beta Q \quad \text{for some} \quad \beta \in \mathbb{R}.$$

First, suppose $\beta = 0$. Then v is an eigenfunction of B with eigenvalue $\alpha < 0$. Now, using more ODE theory (Sturm–Liouville theory), the fact that $BQ' = 0$ and Q' has only one zero then guarantees that v must be positive (the 'ground state' for B). But then $\langle v, Q\rangle = 0$ is impossible, since Q is also positive. We conclude $\beta \neq 0$.

We next consider the function

$$g(\lambda) := \langle (B-\lambda)^{-1}\beta Q, \beta Q\rangle, \quad \lambda \in [\alpha, 0].$$

In particular, by the orthogonality condition, we have

$$g(\alpha) = \beta\langle v, Q\rangle = 0.$$

On the other hand, restricted to $[\mathrm{span}\{Q\}]^\perp$ we can write $B^{-1}(Q) = Z$, so that

$$g(0) = \beta^2\langle Z, Q\rangle = -\tfrac{3}{4}\beta^2\langle Q, Q\rangle < 0.$$

However,

$$g'(\lambda) = \beta^2\|(B-\lambda)^{-1}Q\|_{L^2}^2 > 0.$$

Combining the three displays above yields a contradiction. □

Later we will show that we can choose $x(t)$ to minimize the L^2 or H^1 distance between $u(t)$ and $Q(x - x(t))$, so that

$$\frac{d}{dy}\bigg|_{y=x(t)}\Big[\|u(t) - Q(\cdot - y)\|_X^2\Big] = 0, \quad X \in \{L^2, H^1\}.$$

If we introduce $v(t)$ via

$$u(t,x) = Q(x - x(t)) + v(t, x - x(t)),$$

this minimization condition will guarantee

$$\langle v, Q'\rangle_X = 0.$$

For $X = L^2$, this reads $\langle v, Q'\rangle = 0$, while for $X = H^1$ this reduces (after using the equation for Q) to

$$\langle v, Q'Q\rangle = 0.$$

Our next lemma shows that if this orthogonality condition is *also* satisfied, we can obtain coercivity:

Lemma 10.3. *Let $G \in \{Q', Q'Q\}$. We have*

$$c := \inf\left\{\frac{\langle Bv, v\rangle}{\langle v, v\rangle} : v \in H^1\backslash\{0\} \quad \textit{such that} \quad \langle v, Q\rangle = \langle v, G\rangle = 0\right\} > 0.$$

Proof. By Lemma 10.2, we already know $c \geq 0$. So, suppose $c = 0$ and we will derive a contradiction.

Arguing exactly as in the proof of Lemma 10.2, we can first obtain an L^2-normalized optimizer v. Using the constrained minimization problem satisfied by v (and the assumption $c = 0$), we derive

$$Bv = \alpha Q + \beta G \quad \text{for some} \quad \alpha, \beta \in \mathbb{R}.$$

Recalling $B(Z) = Q$, we see that

$$B(v - \alpha Z) = \beta G.$$

First suppose $G = Q'$. Since $Q' \in N(B) = [R(B)]^\perp$, the identity above implies that $\beta = 0$.

Next, suppose $G = Q'Q$. Since $(Q')^2 Q \geq 0$, we have that $\langle Q'Q, Q' \rangle \neq 0$. Thus $Q'Q \notin [N(B)]^\perp = R(B)$, so in this case we obtain $\beta = 0$ as well.

Thus we have

$$Bv = \alpha Q, \quad \text{i.e.} \quad B(v - \alpha Z) = 0.$$

It follows that $v = \alpha Z + \gamma Q'$ for some $\gamma \in \mathbb{R}$.

Now we apply our orthogonality conditions. First,

$$0 = \langle v, Q \rangle = \alpha \langle Z, Q \rangle + \gamma \langle Q', Q \rangle = -\tfrac{3}{4} \alpha \langle Q, Q \rangle,$$

which forces $\alpha = 0$, i.e. $v = \gamma Q'$. But since Q' is not perpendicular to G (whether $G = Q'$ or $G = QQ'$), we must have $\gamma = 0$ as well.

Thus we derive $v = 0$, contradicting $\|v\|_{L^2} = 1$. □

Now let us prove a slightly weaker version of Theorem 10.1, which requires an additional mass constraint.

Proposition 10.1. *For any $0 < \varepsilon \ll 1$, there exists $\delta > 0$ so that if $u_0 \in H^1$ satisfies*

$$\|u_0 - Q\|_{H^1} < \delta \quad \text{and} \quad M(u_0) = M(Q),$$

then the solution u to KdV with $u|_{t=0} = u_0$ satisfies

$$\sup_{t \geq 0} \|u(t) - Q(\cdot - x(t))\|_{H^1} < \varepsilon$$

for some $x(t) \in \mathbb{R}$. In particular,

$$\sup_{t \geq 0} \inf_{x_0 \in \mathbb{R}} \|u(t) - Q(\cdot - x_0)\|_{H^1} < \varepsilon.$$

Proof, version 1. We wish to define $x(t)$ to as a solution to the minimization problem

$$\text{minimize} \quad a(y) := \|u(t) - Q(\cdot - y)\|_{H^1}^2 \quad \text{over} \quad y \in \mathbb{R}.$$

One can verify that $\lim_{|y| \to \infty} a(y) = \|u(t)\|_{H^1}^2$. We can therefore obtain a minimum and take $x(t)$ to be an argmin. We may also guarantee that $t \mapsto x(t)$ is continuous in time. The condition $a'(y)|_{y = x(t)} = 0$ leads to an orthogonality condition as follows. We define $v(t)$ via

$$u(t, x) = Q(x - x(t)) + v(t, x - x(t)).$$

The condition $a'(y)|_{y = x(t)} = 0$ then reduces to

$$\langle v(t), Q' \rangle_{H^1} = \langle v(t), QQ' \rangle = 0,$$

with $t \mapsto \|v(t)\|_{H^1}$ continuous. Here we have used the equation satisfied by Q. The proof therefore reduces to showing that

$$\|v(t)\|_{H^1} \leq \varepsilon \quad \text{for all} \quad t \geq 0.$$

To this end, let us define

$$I = \{t \geq 0 : \|v(t)\|_{H^1} \leq \varepsilon\},$$

so that it suffices to show $I = [0, \infty)$. Now, by continuity of $t \mapsto \|v(t)\|_{H^1}$, I is closed. We also have $I \ni 0$ for $\delta > 0$ sufficiently small, as

$$\|v(0)\|_{H^1} = \inf_{y \in \mathbb{R}} \|u_0 - Q(\cdot - y)\|_{H^1} \leq \|u_0 - Q\|_{H^1} < \delta.$$

By connectedness of $[0, \infty)$, it therefore suffices to prove that I is also open. This will be done by using the functional L to obtain an *a priori* estimate on v.

To begin, recall from the discussion above that we have

$$L(u_0) - L(Q) \equiv L(u) - L(Q) = \tfrac{1}{2}\langle Bv, v\rangle + \mathcal{O}(\|v\|_{H^1}^3).$$

Now, v satisfies *one* of the orthogonality condition appearing in Lemma 10.3 (namely $\langle v, QQ'\rangle = 0$), but not necessarily the other (i.e. $\langle v, Q\rangle = 0$). It is here that the mass constraint comes into play. Indeed, observe that the constraint $M(u) = M(Q)$ reduces to

$$\langle v, Q\rangle = -\tfrac{1}{2}\langle v, v\rangle.$$

As $v(t)$ is already small, we see that the component of v in the direction of Q is small compared to v itself.

We now decompose

$$v = aQ + w \quad \text{with} \quad \langle w, Q\rangle = 0.$$

In particular,

$$a = \tfrac{\langle v, Q\rangle}{\langle Q, Q\rangle} = \mathcal{O}(\|v\|_{H^1}^2).$$

Note also that since $\langle Q, QQ'\rangle = 0$, we see that $\langle v, QQ'\rangle = 0$ guarantees $\langle w, QQ'\rangle = 0$ as well. Thus w satisfies both orthogonality conditions needed to obtain coercivity.

Thus, as $\|w\|_{H^1} \lesssim \|v\|_{H^1}$, we can use this decomposition of v to obtain

$$\langle Bv, v\rangle = \langle Bw, w\rangle + \mathcal{O}(\|v\|_{H^1}^3 + \|v\|_{H^1}^4).$$

Since w satisfies *both* $\langle w, Q\rangle = 0$ and $\langle w, QQ'\rangle = 0$, Lemma 10.3 applies to w. Applying Lemma 10.1 as well, we find

$$\langle Bv, v\rangle \gtrsim \|w\|_{H^1}^2 + \mathcal{O}(\|v\|_{H^1}^3 + \|v\|_{H^1}^4).$$

Using once again that $a = \mathcal{O}(\|v\|_{H^1}^2)$, we see that

$$\|w\|_{H^1} \gtrsim \|v\|_{H^1}^2 + \mathcal{O}(\|v\|_{H^1}^4),$$

and hence

$$\langle Bv, v\rangle \gtrsim \|v\|_{H^1}^2 + \mathcal{O}(\|v\|_{H^1}^3 + \|v\|_{H^1}^4).$$

Returning to the expansion of L above, we find that there exist $C_1, C_2 > 0$ such that

$$\|v(t)\|_{H^1}^2 \leq C_1|L(u_0) - L(Q)| + C_2(\|v(t)\|_{H^1}^3 + \|v(t)\|_{H^1}^4) \quad \text{for all} \quad t \in I.$$

We can finally show that the set I defined above is open (if $\delta = \delta(\varepsilon)$ is sufficiently small). First, by H^1-continuity of L, we can guarantee that $C_1|L(u_0) - L(Q)| < \frac{1}{2}\varepsilon^2$ for δ sufficiently small. Then, if $t \in I$, the inequality above implies

$$\|v(t)\|_{H^1}^2 \le \tfrac{1}{2}\varepsilon^2 + C_2\varepsilon^2(\tfrac{1}{8}\varepsilon + \tfrac{1}{16}\varepsilon^2) \le \tfrac{3}{4}\varepsilon^2.$$

In particular, by continuity of $t \mapsto \|v(t)\|_{H^1}$, we find that

$$\|v(\tau)\|_{H^1} \le \varepsilon \quad \text{for} \quad \tau \quad \text{in a neighborhood of} \quad t.$$

This shows that I is open, and hence (by connectedness of $[0,\infty)$ we obtain $I = [0,\infty)$, as desired.

□

Proof, version 2. We wish to define $x(t)$ to as a solution to the minimization problem

$$\text{minimize} \quad a(y) := \|u(t) - Q(\cdot - y)\|_{L^2}^2 \quad \text{over} \quad y \in \mathbb{R}.$$

One can verify that $\lim_{|y|\to\infty} a(y) = \|u(t)\|_{L^2}^2$. We can therefore obtain a minimum and take $x(t)$ to be an argmin. We may also guarantee that $t \mapsto x(t)$ is continuous in time. The condition $a'(x(t)) = 0$ leads to an orthogonality condition as follows. We define $v(t)$ via

$$u(t,x) = Q(x - x(t)) + v(t, x - x(t)).$$

The condition $a(x(t)) = 0$ then reduces to

$$\langle v(t), Q'\rangle = 0,$$

with $t \mapsto \|v(t)\|_{H^1}$ continuous. The proof therefore reduces to showing that

$$\|v(t)\|_{H^1} \le \varepsilon \quad \text{for all} \quad t \in \mathbb{R}.$$

Before getting to the heart of the proof, here is a technical claim we need to verify:

$$|x(0)| = o(1) \quad \text{as} \quad \delta \to 0. \tag{10.1}$$

Proof of (10.1). By assumption and definition, we have

$$\|u_0 - Q(\cdot - x(0))\|_{L^2} \le \|u_0 - Q\|_{L^2} < \delta,$$

so that by the triangle inequality we have

$$\|Q - Q(\cdot - x(0))\|_{L^2} \le 2\delta.$$

Thus by explicit calculation (and the fact that we may assume $|x(0)| \le C(u_0)$) we see that $x(0) \to 0$ as $\delta \to 0$. □

Now let us define

$$I = \{t \ge 0 : \|v(t)\|_{H^1} \le \varepsilon\}.$$

To complete the proof, it suffices to show $I = \mathbb{R}$. To this end, first note that (by continuity of $t \mapsto \|v(t)\|_{H^1}$) I is closed. We also claim that $I \ni 0$ (if δ is sufficiently small). Indeed, using (10.1), we have

$$\begin{aligned}
\|v(0)\|_{H^1} &= \|u_0 - Q(\cdot - x(0))\|_{H^1} \\
&\le \|u_0 - Q\|_{H^1} + \|Q - Q(\cdot - x(0))\|_{H^1} \\
&\le \delta + o(1) \quad \text{as} \quad \delta \to 0,
\end{aligned}$$

which implies the claim. By connectedness of $[0,\infty)$, it therefore suffices to prove that I is also open. This will be done by using the functional L to obtain an *a priori* estimate on v.

To begin, recall from the discussion above that we have

$$L(u_0) - L(Q) \equiv L(u) - L(Q) = \tfrac{1}{2}\langle Bv, v\rangle + \mathcal{O}(\|v\|_{H^1}^3).$$

Now, v satisfies *one* of the orthogonality condition appearing in Lemma 10.3 (namely $\langle v, Q'\rangle = 0$), but not necessarily the other (i.e. $\langle v, Q\rangle = 0$). It is here that the mass constraint comes into play. Indeed, observe that the constraint $M(u) = M(Q)$ reduces to

$$\langle v, Q\rangle = -\tfrac{1}{2}\langle v, v\rangle.$$

As $v(t)$ is already small, we see that the component of v in the direction of Q is small compared to v itself. In particular, if we write

$$v = aQ + w, \quad \text{with} \quad \langle w, Q\rangle = 0,$$

then $a = \mathcal{O}(\|v\|_{H^1}^2)$. Note also that $\langle v, Q'\rangle = 0$ guarantees $\langle w, Q'\rangle = 0$, since $\langle Q, Q'\rangle = 0$.

We now write (noting $\|w\|_{H^1} \lesssim \|v\|_{H^1}$),

$$\begin{aligned}\langle Bv, v\rangle &= a^2\langle BQ, Q\rangle + 2a\langle BQ, w\rangle + \langle Bw, w\rangle \\ &= \langle Bw, w\rangle + \mathcal{O}(\|v\|_{H^1}^3 + \|v\|_{H^1}^4).\end{aligned}$$

Since w satisfies *both* $\langle w, Q\rangle = 0$ and $\langle w, Q'\rangle = 0$, Lemma 10.3 applies to w. Applying Lemma 10.1 as well, we find

$$\langle Bv, v\rangle \gtrsim \|w\|_{H^1}^2 + \mathcal{O}(\|v\|_{H^1}^3 + \|v\|_{H^1}^4).$$

Using once again that $a = \mathcal{O}(\|v\|_{H^1}^2)$, we see that

$$\|w\|_{H^1} \gtrsim \|v\|_{H^1}^2 + \mathcal{O}(\|v\|_{H^1}^4),$$

and hence

$$\langle Bv, v\rangle \gtrsim \|v\|_{H^1}^2 + \mathcal{O}(\|v\|_{H^1}^3 + \|v\|_{H^1}^4).$$

Returning to the expansion of L above, we find that there exist $C_1, C_2 > 0$ such that

$$\|v(t)\|_{H^1}^2 \leq C_1|L(u_0) - L(Q)| + C_2(\|v(t)\|_{H^1}^3 + \|v(t)\|_{H^1}^4) \quad \text{for all} \quad t \in I.$$

We can finally show that the set I defined above is open (if $\delta = \delta(\varepsilon)$ is sufficiently small). First, by H^1-continuity of L, we can guarantee that $C_1|L(u_0) - L(Q)| < \frac{1}{2}\varepsilon^2$ for δ sufficiently small. Then, if $t \in I$, the inequality above implies

$$\|v(t)\|_{H^1}^2 \leq \tfrac{1}{2}\varepsilon^2 + C_2\varepsilon^2(\tfrac{1}{8}\varepsilon + \tfrac{1}{16}\varepsilon^2) \leq \tfrac{3}{4}\varepsilon^2.$$

In particular, by continuity of $t \mapsto \|v(t)\|_{H^1}$, we find that

$$\|v(\tau)\|_{H^1} \leq 2\varepsilon \quad \text{for} \quad \tau \quad \text{in a neighborhood of} \quad t.$$

This shows that I is open, and hence (by connectedness of $[0,\infty)$ we obtain $I = [0,\infty)$, as desired. □

To prove Theorem 10.1, it remains to remove the mass constraint in Proposition 10.1. This will rely on the scaling symmetry, as described above.

Proof of Theorem 10.1. Let us first obtain a version of Proposition 10.1 based around an arbitrary Q_c (recall $Q_c(x) = cQ(\sqrt{c}x)$). Let us define

$$\|f\|_{H^1_c}^2 := \|f\|_{L^2}^2 + c^{-1}\|f\|_{\dot{H}^1}^2.$$

We claim that given $c > 0$ and $\varepsilon > 0$, there exists $\delta = \delta(c, \varepsilon) > 0$ such that if

$$\|u_0 - Q_c\|_{H^1_c} < \delta \quad \text{and} \quad \|u_0\|_{L^2} = \|Q_c\|_{L^2},$$

then there exists $x(t) \in \mathbb{R}$ such that

$$\|u(t) - Q_c(\cdot - x(t))\|_{H^1_c} < \varepsilon.$$

To prove this, define

$$w(t, x) = c^{-1}u(c^{-\frac{3}{2}}t, c^{-\frac{1}{2}}x).$$

This is a solution to KdV. Its initial data w_0 satisfies

$$\|w_0\|_{L^2} = c^{-\frac{3}{4}}\|u_0\|_{L^2} = c^{-\frac{3}{4}}\|Q_c\|_{L^2} = \|Q\|_{L^2}.$$

Moreover, by direct calculation,

$$\begin{aligned}\|w_0 - Q\|_{H^1}^2 &= c^{-\frac{3}{2}}\|u_0 - Q_c\|_{L^2}^2 + c^{-\frac{5}{2}}\|u_0 - Q_c\|_{\dot{H}^1}^2 \\ &= c^{-\frac{3}{2}}\|u_0 - Q_c\|_{H^1_c}^2.\end{aligned}$$

Thus for $\delta = \delta(\varepsilon, c)$ sufficiently small, we can apply Proposition 10.1 to the solution w to find $x(t) \in \mathbb{R}$ such that

$$\|w(t) - Q(\cdot - x(t))\|_{H^1}^2 < \varepsilon^2 c^{-\frac{3}{2}} \quad \text{for all} \quad t \geq 0.$$

As

$$\|w(t) - Q(\cdot - x(t))\|_{H^1}^2 = c^{-\frac{3}{2}}\|u(t) - Q_c(\cdot - x(t))\|_{H^1_c}^2,$$

we obtain

$$\|u(t) - Q_c(\cdot - x(t))\|_{H^1_c}^2 < \varepsilon^2,$$

as desired.

Now we return to the original problem. We let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon) > 0$ to be determined more precisely below. We let u_0 satisfy

$$\|u_0 - Q\|_{H^1} < \delta$$

and let u be the KdV solution with data u_0. Now we wish to choose c such that

$$\|u_0\|_{L^2} = \|Q_c\|_{L^2} = c^{\frac{3}{4}}\|Q\|_{L^2}.$$

In particular,

$$c = \Big(\tfrac{\|u_0\|_{L^2}}{\|Q\|_{L^2}}\Big)^{\frac{4}{3}} = 1 + o(1) \quad \text{as} \quad \delta \to 0.$$

We then have

$$\begin{aligned}\|u_0 - Q_c\|_{H^1_c} &\leq \|u_0 - Q\|_{H^1_c} + \|Q - Q_c\|_{H^1_c} \\ &= \|u_0 - Q\|_{H^1} + o(1) \quad \text{as} \quad \delta \to 0,\end{aligned}$$

so that we can apply the rescaled stability result to the solution u. In particular, we can find $x(t) \in \mathbb{R}$ such that

$$\|u(t) - Q_c(\cdot - x(t))\|_{H^1_c} < \varepsilon \quad \text{for all} \quad t \geq 0.$$

But now we observe that for any $t \geq 0$,

$$\begin{aligned}\|u(t) - Q(\cdot - x(t))\|_{H^1} &\leq \|u(t) - Q_c(\cdot - x(t))\|_{H^1} + \|Q - Q_c\|_{H^1} \\ &\leq \|u(t) - Q(\cdot - x(t))\|_{H^1_c} + o(1) \quad \text{as} \quad \delta \to 0 \\ &< \varepsilon + o(1) \quad \text{as} \quad \delta \to 0.\end{aligned}$$

This finally implies the desired orbital stability result! □

11. Scattering and inverse scattering for KdV

Reference: Drazin and Johnson [1].

We follow this reference fairly closely, as the exposition is quite good.

We continue our study of the KdV equation, returning to the formulation

$$u_t - 6uu_x + u_{xxx} = 0$$

for $x \in \mathbb{R}$ and $t > 0$.

We will connect this equation to a *Sturm–Liouville problem* of the form

$$\psi_{xx} + (\lambda - u)\psi = 0,$$

where $u = u(x) \in \mathbb{R}$ is given (later u will be the solution to KdV at a fixed time), λ is a parameter (interpreted as an eigenvalue), and we seek a bounded solution $\psi = \psi(x; \lambda)$.

11.1. Direct scattering.

We assume that $u = u(x)$ is a 'nice' function, for example a Schwartz function (although much less is needed), and we study the problem

$$\psi_{xx} + (\lambda - u)\psi = 0.$$

The *spectrum* (i.e. the possible values of λ) is made up of two types, namely, $\lambda > 0$ or $\lambda \leq 0$. The case $\lambda = 0$ can be ruled out if, for example, $u(x) > 0$ for all x.

Note that as we are assuming $|u| \to 0$ as $|x| \to \infty$, the equation above takes the asymptotic form

$$\psi_{xx} \approx -\lambda\psi \quad \text{as} \quad |x| \to \infty.$$

In the case $\lambda > 0$, it follows that the eigenfunction ψ is asymptotically given as a linear combination of $e^{\pm i\sqrt{\lambda}x}$ as $|x| \to \pm\infty$. We call the case of $\lambda > 0$ the *continuous spectrum.*

In the case $\lambda < 0$, the asymptotic equation at $x = \pm\infty$ now involves linear combinations of $e^{\pm\sqrt{-\lambda}x}$. We are interested in bounded solutions only, so consider for example a real solution satisfying

$$\psi(x) \approx \alpha e^{(-\lambda)^{1/2}x} \quad \text{as} \quad x \to -\infty$$

for some constant α. In general, the corresponding solution will involve both exponential terms as $x \to +\infty$, say

$$\psi(x) \approx \beta e^{(-\lambda)^{\frac{1}{2}}x} + \gamma e^{-(-\lambda)^{\frac{1}{2}}x} \quad \text{as} \quad x \to \infty,$$

with $\beta, \gamma \in \mathbb{R}$. This solution is not bounded unless $\beta = 0$. The values of λ for which we obtain $\beta = 0$ are then called the *discrete spectrum* of eigenvalues, and the corresponding eigenfunctions decay exponentially as $|x| \to \infty$.

Given $u(x)$, there may be no discrete spectrum at all. For example, if $u > 0$, then by taking an inner product of the eigenvalue problem with ψ and integrating by parts, we obtain

$$-\|\psi_x\|_{L^2}^2 + \lambda\|\psi\|_{L^2}^2 = \int u(x)\psi^2(x)\,dx.$$

If $\lambda < 0$ but $u > 0$, we obtain a contradiction.

If $u \leq 0$ and $u \to 0$ sufficiently rapidly as $|x| \to \infty$, one can show that the number of discrete eigenvalues is necessarily *finite*. We won't pursue the general theory here. We will also always assume that we are dealing with simple eigenvalues.

We next collect several properties of eigenfunctions. For example, we can show that eigenfunctions are bounded and continuous, and typically at least once differentiable. For example, integrating the equation yields

$$\psi_x(b) - \psi_x(a) = \int_a^b (u - \lambda)\psi\,dx,$$

which (sending $b \to a$) yields continuity of ψ_x provided the integral tends to zero. This is usually the case (and may hold even for discontinuous u, but would fail if e.g. $u = \delta_0$).

Similarly, integrating first from c to x and then from a to b leads to

$$\psi(b) - \psi(a) = (b-a)\psi_x(c) + \int_a^b \int_c^x (u - \lambda)\psi\,dy\,dx,$$

yielding continuity of ψ (even if, say $u = \delta_0$).

While all of our eigenfunctions will be bounded and continuous, only the discrete eigenfunctions will decay at $\pm\infty$. In particular, discrete eigenfunctions are guaranteed to belong to L^2.

For the discrete spectrum, let us define

$$\kappa_n = (-\lambda_n)^{\frac{1}{2}}, \quad n = 1, \ldots, N,$$

ordered via $\kappa_1 < \cdots < \kappa_N$. We will fix an eigenfunction satisfying

$$\psi_n(x) \approx c_n \exp(-\kappa_n x) \quad \text{as} \quad x \to \infty,$$

where the *norming constant* c_n is fixed by imposing the L^2-normalization

$$\int_{\mathbb{R}} \psi_n^2\,dx = 1.$$

For the continuous spectrum, let us denote $k = \pm\sqrt{\lambda}$ and define a solution given by the following behavior at $\pm\infty$:

$$\hat{\psi}(x;k) \approx \begin{cases} e^{-ikx} + be^{ikx} & x \to +\infty, \\ ae^{-ikx} & x \to -\infty. \end{cases}$$

Now, given two solutions ψ, φ satisfying

$$\psi_{xx} + (\lambda - u)\psi = 0 \quad \text{and} \quad \varphi_{xx} + (\mu - u)\varphi = 0,$$

we define the *Wronskian* $W(\psi, \varphi)$ via

$$W(\psi, \varphi) = \psi\varphi_x - \psi_x\varphi.$$

Then

$$\tfrac{d}{dx} W(\psi, \varphi) = (\lambda - \mu)\psi\varphi.$$

From this, it follows that (i) eigenfunctions corresponding to distinct (discrete) eigenvalues are orthogonal (with respect to the L^2-norm) and (ii) if $\lambda = \mu$ then $W(\psi,\varphi) \equiv \text{const}$). (It is also true that any continuous eigenfunction is orthogonal to any discrete eigenfunction.)

Given the continuous eigenfunction $\hat\psi(x,k)$, its complex conjugate $\hat\psi^*(x,k)$ is also a continuous eigenfunction with parameter k, so that $W(\psi,\psi^*)$ is constant. Evaluating as $x \to \pm\infty$, we obtain

$$W(\hat\psi,\hat\psi^*) = 2ikaa^* = 2ik(1-bb^*), \quad \text{i.e.} \quad |a|^2 + |b|^2 = 1.$$

In the setting of quantum mechanics, the function $u(x)$ is an 'external potential', and we interpret the discrete eigenfunctions as the *bound states* (or stationary states). The continuous eigenfunctions are asymptotically free waves. They are not themselves admissible quantum states (they are not in L^2), but they can be combined into 'wave packets'. In fact, it is a general fact that the discrete and continuous eigenfunctions together form a *complete* set. This means that any $f \in L^2$ can be written in the form

$$f(x) = \sum_{n=1}^N \alpha_n \psi_n(x) + \int_{\mathbb{R}} \alpha(k)\hat\psi(x;k)\,dk.$$

The eigenfunction $\hat\psi(x;k)$ is interpreted as an incident wave from $x = +\infty$ (i.e. e^{-ikx}), together with a reflected wave to $x = +\infty$ (be^{ikx}) and a transmitted wave at $x = -\infty$ (ae^{ikx}). The constants $a = a(k)$ and $b = b(k)$ are called the *transmission* and *reflection* coefficients. The relation $|a|^2 + |b|^2 = 1$ is a statement of the conservation of energy. We have generally that in the limit $|k| \to \infty$, we get $a(k) \to 1$ and $b(k) \to 0$.

We turn to some explicit examples.

Example 25 (Dirac delta). *We begin by considering*

$$u(x) = -U_0\delta(x), \quad U_0 > 0.$$

To begin, we need to clarify the behavior of solutions at $x = 0$. If we integrate

$$\psi_{xx} + (\lambda + U_0\delta)\psi = 0$$

from $-\varepsilon$ to ε, we get

$$\psi'(\varepsilon) - \psi'(-\varepsilon) + \int_{-\varepsilon}^{\varepsilon} [\lambda + U_0\delta(x)]\psi(x)\,dx = 0,$$

so that sending $\varepsilon \to 0$ yields

$$[\psi'] = -U_0\psi(0), \quad [f] := f(0+) - f(0-).$$

(One can verify with a second integration that ψ is continuous for all $x \in \mathbb{R}$.)

Now, $u = 0$ for $x < 0$ or $x > 0$. Thus, if we consider a (discrete) eigenvalue $\kappa_n = (-\lambda)^{\frac12}$, we must have

$$\psi_n(x) = \begin{cases} \alpha_n \exp(-\kappa_n x) & x > 0 \\ \beta_n \exp(\kappa_n x) & x < 0. \end{cases}$$

To obtain continuity at $x = 0$, we must have $\alpha_n = \beta_n$. The L^2 normalization of ψ_n requires

$$\alpha_n^2 \left[\int_{-\infty}^0 \exp(2\kappa_n x)\,dx + \int_0^\infty \exp(-2\kappa_n x)\,dx \right] = 1,$$

which (choosing $\alpha_n > 0$) requires $\alpha_n = (\kappa_n)^{\frac{1}{2}}$. It remains to impose the jump condition:

$$[\psi_n'] = -\kappa_n\alpha_n - \kappa_n\alpha_n = -U_0\alpha_n, \quad \text{so that} \quad \kappa_n = \tfrac{1}{2}U_0.$$

It follows that there is exactly one eigenvalue $\lambda_1 = -\frac{1}{4}U_0^2$ (with $\kappa_1 = \frac{1}{2}U_0$). The eigenfunction is

$$\psi(x) = (\tfrac{1}{2}U_0)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}U_0|x|\}.$$

Note that no eigenvalue would occur if we had taken $U_0 < 0$.

How about the continuous eigenfunctions? Recalling the definition of $\hat{\psi}(x;k)$ and the fact that $\delta = 0$ for $x \neq 0$, we have

$$\hat{\psi}(x;k) = \begin{cases} e^{-ikx} + b(k)e^{ikx} & x > 0 \\ a(k)e^{-ikx} & x < 0. \end{cases}$$

For continuity at $x = 0$, we need $a = 1 + b$. The jump condition at $x = 0$ is written

$$[\hat{\psi}'] = -ik + bik - (-ika) = -U_0\psi(0) = -U_0(1+b),$$

which implies

$$b(k) = -\frac{U_0}{U_0 + 2ik}, \quad a(k) = \frac{2ik}{U_0 + 2ik}.$$

In this case we find that the continuous spectrum exists whether U_0 is positive or negative. The pole in the upper half-plane of $b(k), a(k)$ corresponds to the discrete eigenvalue if k is extended into the complex plane (that is, the pole is at $k = i\frac{1}{2}U_0$, i.e. $\lambda = -\frac{1}{4}U_0^2$). It is a general fact that the discrete eigenvalues can be determined from the eigenfunction for the continuous spectrum (more precisely, its extension into a half-plane).

As an exercise you may try to work out the case $u = U_0\chi_{(0,1)}$. The computations are more involved.

Example 26 (Soliton)**.** *Next, consider*

$$u(x) = -U_0 \operatorname{sech}^2 x, \quad U_0 \in \mathbb{R}\backslash\{0\}.$$

The eigenvalue problem is then

$$\psi_{xx} + (\lambda + U_0 \operatorname{sech}^2 x)\psi = 0$$

We introduce $T = \tanh(x)$. Then $x \in \mathbb{R}$ corresponds to $T \in (-1,1)$, and we have

$$\tfrac{dT}{dx} = \operatorname{sech}^2(x) = 1 - \tanh^2(x) = 1 - T^2.$$

Writing $\psi(x) = \Psi(T)$ and noting

$$\tfrac{d}{dx} = (1 - T^2)\tfrac{d}{dT},$$

the equation reduces to

$$\tfrac{d}{dT}\big\{(1-T^2)\tfrac{d\Psi}{dT}\big\} + (U_0 + \tfrac{\lambda}{1-T^2})\Psi = 0.$$

This is a well-known 2nd order ODE known as the associated Legendre equation*.*

We can obtain bounded solutions for $T \in [-1,1]$ only when

$$U_0 = N(N+1), \qquad \lambda_n = -n^2,$$

where $N \in \mathbb{N}$ and $n = 1, \ldots, N$. There is a family of solutions P_N^n, known as the associated Legendre functions*, which have the form*

$$P_N^n(T) = (-1)^n(1-T^2)^{\frac{n}{2}}(\tfrac{d}{dT})^n P_N(T), \quad \text{with} \quad P_N(T) = \tfrac{1}{N!2^N}(\tfrac{d}{dT})^N(T^2-1)^N.$$

We take eigenfunctions to be proportional to the P_N^n, with norming constants to impose L^2-normalization. Some specific examples include

$$\psi_1(x) = (\tfrac{3}{2})^{\frac{1}{2}} \tanh x \operatorname{sech} x,$$
$$\psi_2(x) = \tfrac{3^{\frac{1}{2}}}{2} \operatorname{sech}^2 x,$$

corresponding to eigenvalues $\lambda_1 = -1$, $\lambda_2 = -4$.

For the continuous spectrum, we return to general $U_0 \in \mathbb{R}\backslash\{0\}$ and have $\lambda = k^2 > 0$. In this case, the associated Legendre equation can be solved using the hypergeometric function $F(\alpha, \beta; \gamma, z)$. Imposing the desired boundary condition as $x \to -\infty$, the result is

$$\hat{\psi}(x; k) = a(k) 2^{ik} [\operatorname{sech} x]^{-ik} F(\alpha, \beta; \gamma, z),$$

where

$$\begin{aligned} \alpha &= \tfrac{1}{2} - ik + (U_0 + \tfrac{1}{4})^{\frac{1}{2}}, \\ \beta &= \tfrac{1}{2} - ik - (U_0 + \tfrac{1}{4})^{\frac{1}{2}}, \\ \gamma &= 1 - ik, \\ z &= \tfrac{1}{2}(1 + T). \end{aligned}$$

We have $z \to 0^+$ as $x \to -\infty$ and $z \to 1^-$ as $x \to +\infty$. One can then verify

$$\hat{\psi}(x; k) \sim a(k) e^{-ikx} \quad as \quad x \to -\infty.$$

Furthermore, as $x \to +\infty$, one has

$$\hat{\psi}(x; k) \approx \tfrac{a(k)\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} e^{-ikx} + \tfrac{a(k)\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} e^{ikx},$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ is the Gamma function. For the boundary conditions at $\pm\infty$ to be satisfied, we must therefore have

$$a(k) = \tfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)} \quad and \quad b(k) = \tfrac{a(k)\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

which determines the scattering coefficients as functions of k and U_0.

Remark 1. *Observe that there are certain values of U_0 such that $b(k) \equiv 0$ (yielding so-called 'reflectionless' potentials). To see this, we first recall the following Gamma function identity:*

$$\Gamma(\tfrac{1}{2} - z)\Gamma(\tfrac{1}{2} + z) = \tfrac{\pi}{\cos \pi z},$$

so that

$$\begin{aligned} \Gamma(\gamma - \alpha)\Gamma(\gamma - \beta) &= \Gamma(\tfrac{1}{2} - (U_0 + \tfrac{1}{4})^{\frac{1}{2}})\Gamma(\tfrac{1}{2} + (U_0 + \tfrac{1}{4})^{\frac{1}{2}}) \\ &= \frac{\pi}{\cos(\pi(U_0 + \frac{1}{4})^{\frac{1}{2}})}. \end{aligned}$$

Thus we obtain $b(k) \equiv 0$ if

$$(U_0 + \tfrac{1}{4})^{\frac{1}{2}} = N + \tfrac{1}{2}, \quad i.e. \quad U_0 = N(N+1), \quad N = 1, 2, \ldots.$$

In particular, the discrete eigenfunctions obtained above are reflectionless potentials.

Remark 2. *Although we won't prove it here, one can verify that for $U_0 > 0$, one will have a finite number of eigenvalues, while if $U_0 < 0$ only continuous spectrum is present. This can be worked out by considering the poles of the functions $a(k), b(k)$ introduced above.*

11.2. **Inverse Scattering.** In the previous section, we described how a given potential $u = u(x)$ determines certain *scattering data* related to the problem

$$\psi_{xx} + (\lambda - u)\psi = 0, \quad x \in \mathbb{R}.$$

This includes:

- Discrete spectral data: this includes the eigenvalues $\lambda_n < 0$ and norming constants c_n.

  This means there L^2-normalized eigenfunctions ψ_n satisfying

  $$\psi_n(x) \approx c_n \exp(-\kappa_n x) \quad \text{as} \quad x \to +\infty, \quad \kappa_n = (-\lambda_n)^{1/2}.$$

- Continuous spectral data:, this includes the transmission and reflection coefficients a, b, which we view as a function of $k = \pm\lambda^{1/2}$, with $\lambda > 0$.

  This means that we have continuous eigenfunctions $\hat{\psi}(x;k)$ satisfying

  $$\hat{\psi}(x;k) \approx \begin{cases} e^{-ikx} + b(k)e^{ikx} & x \to +\infty, \\ a(k)e^{-ikx} & x \to -\infty. \end{cases}$$

The inverse scattering problem asks whether knowledge of the scattering data (i.e. $\lambda_n, c_n, a(k), b(k)$) suffices to determine (or 'recover', or 'reconstruct') the potential u.

Let us try to give the general idea here. Because we consider potentials u that decay at infinity, the equation $\psi_{xx} + (\lambda - u)\psi = 0$ may be viewed as a perturbation of of the equation $\psi_{xx} + \lambda\psi = 0$ near $x = +\infty$. This latter equation has the plane wave solution e^{ikx}, so we may look for a solution that has this form asymptotically as $x \to \infty$. For example, we may look for a solution of the form

$$\psi(x;k) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}\,dz. \tag{11.1}$$

We might also allow the *kernel* K to depend on k, but this turns out not to be needed. Presumably, if we can figure out how to define K, it will carry some information about the potential u. Ultimately, to solve the inverse scattering problem, we will need to figure out how to determine K from the scattering data alone.

Let us see what conditions on $K = K(x,z)$ guarantee that ψ solves $\psi_{xx} + (k^2 - u)\psi = 0$.

Assume ψ has the form above. First, apply the fundamental theorem of calculus to write:

$$\begin{aligned}\psi_{xx} = &-[k^2 + 2K_x(x,x) + K_z(x,x) + ikK(x,x)]e^{ikx} \\ &+ \int_x^\infty K_{xx}(x,z)e^{ikz}\,dz.\end{aligned}$$

For ψ itself, it will be useful to use the identity

$$e^{ikz} = \tfrac{1}{ik}\tfrac{d}{dz}e^{ikz}$$

and integrate by parts (twice). This implies

$$\begin{aligned}\psi = [1 &- \tfrac{1}{ik}K(x,x) - \tfrac{1}{k^2}K_z(x,x)]e^{ikx} \\ &- \tfrac{1}{k^2}\int_x^\infty K_{zz}(x,z)e^{ikz}\,dz,\end{aligned}$$

so that

$$k^2\psi = [k^2 + ikK(x,x) - K_z(x,x)]e^{ikx} - \int_x^\infty K_{zz}(x,z)e^{ikz}\,dz.$$

Thus

$$\begin{aligned}\psi_{xx} + k^2\psi - u\psi = -e^{ikx}\big\{2[K_x(x,x) + K_z(x,x)] + u(x)\big\}\\ + \int_x^\infty [K_{xx} - K_{zz} - u(x)K]e^{ikz}\,dz.\end{aligned}$$

For ψ to be a solution, we therefore need

$$\begin{aligned}&K_x(x,x) + K_z(x,x) = -\tfrac{1}{2}u(x),\\ &K_{xx} - K_{zz} - uK = 0 \quad \text{for} \quad z > x,\end{aligned} \tag{11.2}$$

with the boundary conditions $K, K_z \to 0$ as $z \to \infty$. Note that these conditions confirm that we can assume K is independent of k, and also show that we can take K to be real-valued (since u is).

So far we have essentially seen how u could determine K. Instead, we need to see how knowledge of the scattering data could determine the kernel K (and then use the first equation in (11.2) to determine u). The reflection coefficient b is involved in defining the continuous eigenfunction $\hat\psi$, which has different asymptotic behavior than the solution ψ we just considered. In particular, by considering the behavior as $x \to +\infty$, we must have

$$\hat\psi = \psi^* + b(k)\psi.$$

Using this, let us try to connect the unknown $K(x,z)$ with $b(k)$. Using $K(x,z) = 0$ for $z < x$, we may write

$$\hat\psi(x;k) = e^{-ikx} + b(k)e^{ikx} + \int_{\mathbb{R}} K(x,z)e^{-ikz}\,dz + b(k)\int_{\mathbb{R}} K(x,z)e^{ikz}\,dz.$$

Rearranging leads to

$$\int_{\mathbb{R}} K(x,z)e^{-ikz}\,dz = \hat\psi(x;k) - e^{-ikx} - b(k)e^{ikx} - b(k)\int_{\mathbb{R}} K(x,z)e^{ikz}\,dz.$$

We view this as a formula prescribing (implicitly) the Fourier transform of the function $K(x,\cdot)$ at k. Applying the inverse Fourier transform,

$$K(x,z) = \tfrac{1}{2\pi}\int_{\mathbb{R}} \Big\{\hat\psi(x;k) - e^{-ikx} - b(k)e^{ikx} - b(k)\int_{\mathbb{R}} K(x,y)e^{iky}\,dy\Big\}e^{ikz}\,dk.$$

Now let us introduce

$$F_0(x) := \tfrac{1}{2\pi}\int_{\mathbb{R}} b(k)e^{ikx}\,dk.$$

Then the equation for K above can be rewritten

$$K(x,z) = \tfrac{1}{2\pi}\int_{\mathbb{R}} \big\{\hat\psi(x;k) - e^{-ikx}\big\}e^{ikz}\,dz - F_0(x+z) - \int_{\mathbb{R}} K(x,y)F_0(y+z)\,dy. \tag{11.3}$$

Now let us first consider the simpler scenario in which there is **no discrete spectrum**. This setting, the continuous eigenfunction $\hat\psi(x;k)$ has no poles in the upper half-plane, and we have $|\hat\psi e^{ikx}| \to 1$ as $|k| \to \infty$. Using Cauchy's theorem (on a semicircular contour in the upper half-plane), we can derive that

$$\int_{\mathbb{R}} \{\hat\psi(x,k) - e^{-ikx}\}e^{ikz}\,dz \equiv 0.$$

In this case, the equation for K above becomes

$$\begin{aligned} &K(x,z) + F_0(x+z) + \int_x^\infty K(x,y)F_0(y+z)\,dy = 0, \quad x < z, \\ &F_0(y) = \tfrac{1}{2\pi}\int_{\mathbb{R}} b(k)e^{iky}\,dk. \end{aligned} \tag{11.4}$$

This is called the *Gel'fand–Levitan equation* or *Marchenko equation*. We can iteratively construct a unique solution (more on this below) to obtain the kernel $K(x,z)$, given purely in terms of F (the inverse Fourier transform of the reflection coefficient). Then, as in (11.2), we can recover the potential u from knowledge of K.

Now consider the **general case** (including possible discrete spectrum), returning to (11.3). In this case, the integral no longer vanishes, and instead we have from the residue theorem that

$$\tfrac{1}{2\pi}\int_{\mathbb{R}} \{\hat{\psi}(x;k) - e^{-ikx}\}e^{ikz}\,dz = i\sum_{n=1}^N R_n$$

where R_n is the residue of $\hat{\psi}(x;k)e^{ikz}$ at $k = i\kappa_n$ (where $\lambda_n = -\kappa_n^2$ are the eigenvalues). We turn now to the problem of determining the residues R_n:

Henceforth we denote the solution ψ obtained above by ψ_+ (as its asymptotics at $x \to \infty$ are prescribed), and we introduce a second solution

$$\psi_-(x;k) = e^{-ikx} + \int_{-\infty}^x L(x,z)e^{-ikz}\,dz$$

that is asymptotic to e^{-ikx} as $x \to -\infty$. In particular we can write

$$\hat{\psi}(x;k) = a(k)\psi_- = \psi_+^* + b(k)\psi_+,$$

so that

$$\psi_- = \tfrac{1}{a}\psi_+^* + \tfrac{b}{a}\psi_+.$$

Note that both $\psi_\pm$ are defined at $k = i\kappa_n$. In fact, we have

$$\psi_\pm(x;i\kappa_n) \sim e^{-\kappa_n|x|} \quad \text{as} \quad x \to \pm\infty.$$

Thus, recalling that ψ_n denotes the n^{th} normalized eigenfunction, we can write

$$\psi_n(x) = c_n\psi_+(x;i\kappa_n) = d_n\psi_-(x;i\kappa_n) \tag{11.5}$$

where c_n is the norming constant and $d_n \in \mathbb{R}$ is some (unknown) real constant that we will need to carry around in the analysis.

Now, as the Wronskian $W(\psi_+,\psi_-)$ is independent of x, we can evaluate it by sending $x \to \infty$ and using the formula relating ψ_+ and ψ_- above. In particular:

$$W(\psi_+,\psi_-) = 2ik\tfrac{1}{a(k)}.$$

Note that this identity may be used to derive the following important facts: a^{-1} has zeros at $k = i\kappa_n$, so that a has poles at these points.

Now, differentiate this Wronskian identity with respect to k. This yields

$$W(\partial_k\psi_-,\psi_+) + W(\psi_-,\partial_k\psi_+) = \tfrac{2i}{a}[1 - k\tfrac{a'}{a}], \tag{11.6}$$

which can further be used to show that a has only simple poles.

To get information about $\partial_k\psi_\pm$, we differentiate the equation $\psi_{xx}+(k^2-u)\psi=0$ with respect to k and multiply by ψ to derive the general formula

$$W(\partial_k\psi,\psi)(x) = 2k\int^x \psi^2\,dx. \tag{11.7}$$

In particular, using $W^\pm$ to denote the limit as $x\to\pm\infty$, we obtain

$$W^+(\partial_k\psi_+,\psi_+)\big|_{i\kappa_n} - W^-(\partial_k\psi_+,\psi_+)\big|_{i\kappa_n} = 2i\kappa_n\int_{\mathbb{R}}\psi_+^2\,dx = 2i\tfrac{\kappa_n}{c_n^2}.$$

On the other hand, using (11.6), (11.5), and the fact that $\frac{1}{a}$ vanishes at $i\kappa_n$, we find

$$W^-(\partial_k\psi_-,\tfrac{d_n}{c_n}\psi_-)|_{i\kappa_n} + W^-(\tfrac{c_n}{d_n}\psi_+,\partial_k\psi_+)|_{i\kappa_n} = 2\kappa_n(\tfrac{a'}{a^2})|_{i\kappa_n}.$$

We would now like to combine the previous two displayers to eliminate the $W^-(\partial_k\psi_+,\psi_+)$ term (the value of which is unknown). Using $W(af,g)=aW(f,g)$ and $W(f,g)=-W(g,f)$ and multiplying the second display by $-\frac{d_n}{c_n}$, we find

$$W^+(\partial_k\psi_+,\psi_+)|_{i\kappa_n} - [\tfrac{d_n}{c_n}]^2W^-(\partial_k\psi_-,\psi_-)|_{i\kappa_n} = 2i\tfrac{\kappa_n}{c_n^2} - 2\tfrac{d_n}{c_n}\kappa_n(\tfrac{a'}{a^2})|_{i\kappa_n}.$$

Using (11.7) and the fact that $c_n\psi_+=d_n\psi_-$ at $k=i\kappa_n$, we find that the left-hand side equals zero! Thus we derive

$$(\tfrac{a'}{a^2})|_{i\kappa_n} = \tfrac{i}{c_nd_n}.$$

Now, recalling $\hat\psi = a\psi_-$, we can write

$$\hat\psi(x;k)e^{ikz} = a(k)\psi_-(x;k)e^{ikz} = \tfrac{c_n}{d_n}a(k)\psi_+(x;k)e^{ikz},$$

with only $a(k)$ having poles (for the right-most function). Thus, using the identity above and the fact that a has only simple poles, the residue at $k=i\kappa_n$ is

$$\begin{aligned} R_n &= \tfrac{c_n}{d_n}\psi_+(x;i\kappa_n)e^{-\kappa_n z}\operatorname{res}|_{k=i\kappa_n}a \\ &= -\tfrac{c_n}{d_n}\psi_+(x;i\kappa_n)e^{-\kappa_n z}(\tfrac{a^2}{a'})|_{i\kappa_n} \\ &= ic_n^2\psi_+(x;i\kappa_n)e^{-\kappa_n z}. \end{aligned}$$

For the value $\psi_+(x;i\kappa_n)$, we go back to the defining equation (11.1). The conclusion is:

$$\begin{aligned} &\tfrac{1}{2\pi}\int_{\mathbb{R}}[\hat\psi(x;k)-e^{-ikx}]e^{ikz}\,dz \\ &\qquad = -\sum_{n=1}^N c_n^2e^{-\kappa_n z}\Big\{e^{-\kappa_n x} + \int_x^\infty K(x,y)e^{-\kappa_n y}\,dy\Big\}. \end{aligned}$$

Thus, in the presence of discrete spectrum, (11.3) becomes the Marchenko equation

$$K(x,z)+F(x+z)+\int_x^\infty K(x,y)F(y+z)\,dy = 0,$$

$$F(y) := \sum_{n=1}^N c_n^2e^{-\kappa_n y} + \tfrac{1}{2\pi}\int_{\mathbb{R}} b(k)e^{iky}\,dk,$$

which extends the case without discrete spectrum.

This completes the formulation of the inverse scattering problem. To summarize, the Marchenko equation shows to determine the kernel K in terms of the scattering data $\{b(k),c_n,\kappa_n\}$. The potential u is then reecovered via (11.2). To close this section, let us briefly point out that a practical method for solving the Marchenko equation is via iteration.

11.3. **Solving KdV using inverse scattering.** In this section, we show the direct and inverse scattering problems introduced above may be used to produce solutions to the KdV equation.

We suppose $u = u(t, x)$ is a solution to

$$u_t + u_{xxx} - 6uu_x = 0, \quad u|_{t=0} = u_0.$$

For each $t \geq 0$, we let $u(t)$ be the potential in the Sturm–Liouville problem

$$\psi_{xx} + (\lambda - u)\psi = 0,$$

where now $u = u(t)$ and $\lambda = \lambda(t)$. In the previous sections we developed the maps between the potential u and its scattering data, say S. If we can determine the time evolution of the scattering data (assuming $u(t)$ solves KdV), then we obtain a potential method for producing the solution to KdV at all later names, namely:

- $u_0 \mapsto S_0$ (direct scattering map)
- $S_0 \mapsto S(t)$ (evolution of scattering data)
- $S(t) \mapsto u(t)$ (inverse scattering map).

This is a nonlinear analogue of the Fourier transform method we previously used for linear dispersive PDE, in which the direct/inverse scattering map are replaced by the Fourier transform and its inverse, and the time evolution of the Fourier transform is determined purely by the dispersion relation.

We turn to the problem of determining the time evolution of the scattering data.

Lemma 11.1. *Let u solve KdV and ψ solve $\psi_{xx} + (\lambda - u)\psi = 0$. Define*

$$R(t, x) = \psi_t + u_x\psi - 2(u + 2\lambda)\psi_x.$$

Then

$$\partial_x[\psi_x R - \psi R_x] = \lambda_t\psi^2,$$

where subscripts denote derivatives.

Remark 11.1. *I wish I had an insightful way to motivate the introduction of R. Unfortunately, I do not. If you are reading these notes and know how one might naturally 'discover' R, please let me know!.*

Proof. Start by differentiating the Sturm–Liouville equation with respect to t and x:

$$\begin{aligned}
\psi_{xxt} + (\lambda_t - u_t)\psi + (\lambda - u)\psi_t &= 0, \\
\psi_{xxx} - u_x\psi + (\lambda - u)\psi_x &= 0.
\end{aligned}$$

By direct calculation,

$$\begin{aligned}
\partial_x[\psi_x R - \psi R_x] &= \psi_{xx}R - \psi R_{xx} \\
&= \psi_{xx}[\psi_t + u_x\psi - 2(u + 2\lambda)\psi_x] \\
&\quad - \psi[\psi_{xxt} + u_{xxx}\psi - 3u_x\psi_{xx} - 2(u + 2\lambda)\psi_{xxx}].
\end{aligned}$$

We can now insert the identities for ψ_{xxt} and ψ_{xxx} above and use the equation for ψ to replace instances of ψ_{xx} with $-(\lambda - u)\psi$. After tedious algebraic manipulations, one finally arrives at

$$\partial_x(\psi_x R - \psi R_x) = \psi^2(\lambda_t - u_t - u_{xxx} + 6uu_x),$$

which (since u solves KdV) yields the result. $\square$

We first consider the case of **discrete spectrum**, so that $\lambda|_{t=0} = -\kappa_n^2$ with $\psi|_{t=0} = \psi_n$. Integrate the result of the lemma over $x \in \mathbb{R}$ (writing R_n for the function R above defined using ψ_n). Using the decay of ψ and R, this yields

$$0 = -\partial_t(\kappa_n)^2 \int_{\mathbb{R}} \psi_n^2 \, dx.$$

Thus κ_n is constant, i.e. the discrete eigenvalues are conserved in time!

We next compute the time evolution of the norming constants c_n. First, integrating the result of the lemma and using decay at infinity, we obtain

$$\psi_{nx} R_n - \psi_n R_{nx} \equiv 0$$

Integrating once more leads to

$$\tfrac{R_n}{\psi_n} = h_n$$

for some function $h_n = h_n(t)$, so that (recalling the definition of R_n)

$$\psi_n[\partial_t \psi_n + u_x \psi_n - 2(u - 2\kappa_n^2)\psi_{nx}] = h_n \psi_n^2. \tag{11.8}$$

Now, multiplying the equation for ψ_n by ψ_{nx} implies

$$u\psi_n\psi_{nx} = \psi_{nxx}\psi_{nx} - \kappa_n^2 \psi_n \psi_{nx}.$$

Writing $u_x\psi_n^2 = (u\psi_n^2)_x - 2u\psi_n\psi_{nx}$, we therefore obtain

$$\begin{aligned} &u_x\psi_n^2 - 2u\psi_n\psi_{nx} + 4\kappa_n^2\psi_n\psi_{nx} \\ &\quad = (u\psi_n^2)_x - 4u\psi_n\psi_{nx} + 4\kappa_n^2\psi_n\psi_{nx} \\ &\quad = (u\psi_n^2)_x - 4\psi_{xx}\psi_x + 8\kappa_n^2\psi_n\psi_{nx}. \end{aligned}$$

Thus (11.8) can be written

$$\tfrac{1}{2}(\psi_n^2)_t + [u\psi_n^2 - 2\psi_{nx}^2 + 4\kappa_n^2\psi_n^2]_x = h_n\psi_n^2.$$

We can therefore integrate over $x \in \mathbb{R}$ (observing limits are independent of t) to obtain

$$\tfrac{1}{2}\tfrac{d}{dt}\int \psi_n^2 \, dx = h_n \int \psi_n^2 \, dx,$$

which (by normalization) forces $h_n \equiv 0$. It follows that

$$R_n = \psi_{nt} + u_x\psi_n - 2(u - 2\kappa_n^2)\psi_{nx} = 0,$$

which we view as prescribing the time evolution for ψ_n. Since $\psi_n(t,x) \sim c_n(t)e^{-\kappa_n x}$ as $x \to \infty$, such evolution implies that

$$\tfrac{dc_n}{dt} = 4\kappa_n^3 c_n, \quad \text{i.e.} \quad c_n(t) = c_n(0)e^{4\kappa_n^3 t}.$$

Thus we obtain the time evolution of the norming constants.

We turn to the **continuous spectrum**, for which $\lambda = k^2 > 0$, where $k \in \mathbb{R}$ can take on any value. We are then going to consider continuous eigenfunctions $\hat\psi = \hat\psi(t,x;k)$, i.e. with k fixed but u evolving. Applying the lemma with $\hat\psi$ (and $\lambda_t = 0$) and denoting $\hat R$ the function above evaluated with $\hat\psi$, we integrate to find

$$\hat\psi_x \hat R - \hat\psi \hat R_x = g(t;k)$$

for some g. Now,

$$\hat\psi(t,x;k) \sim a(t;k)e^{-ikx} \quad \text{as} \quad x \to -\infty,$$

so that by the definition of R we obtain

$$\hat{R}(t,x;k) \sim \left[\tfrac{da}{dt} + 4ik^3 a\right] e^{-ikx} \quad \text{as} \quad x \to -\infty.$$

It follows that $\hat{\psi}_x \hat{R} - \hat{\psi}\hat{R}_x \to 0$ as $x \to -\infty$, whence $g \equiv 0$. We therefore integrate again (as above) to obtain

$$\hat{R} = h(t;k)\hat{\psi}$$

for some h. Inserting the asymptotic behavior of $\hat{\psi}, \hat{R}$ as $x \to -\infty$, we obtain

$$\tfrac{da}{dt} + 4ik^3 a = ha.$$

On the other hand, as $x \to \infty$,

$$\begin{aligned} \hat{R}(t,x;k) &\sim \tfrac{db}{dt} e^{ikx} + 4ik^3(e^{-ikx} - be^{ikx}), \\ \hat{\psi}(t,x;k) &\sim e^{-ikx} + b(t;k)e^{ikx}, \end{aligned}$$

so that we obtain

$$\tfrac{db}{dt} e^{ikx} + 4ik^3(e^{-ikx} - be^{ikx}) = h(e^{-ikx} + be^{ikx}).$$

Using the fact that $e^{\pm ikx}$ are linearly independent, we derive

$$\tfrac{db}{dt} - 4ik^3 b = hb \quad \text{and} \quad h(t;k) = 4ik^3.$$

Then our equations above reduce to

$$a(t;k) \equiv a(0;k) \quad \text{and} \quad b(t;k) = b(0;k)e^{8ik^3 t}.$$

As we now understand completely the evolution of the scattering data, we can carry out our scheme above to obtain the KdV solution at all times $t \geq 0$!

Theorem 11.1. *Suppose u solves $u_t + u_{xxx} - 6uu_x = 0$ with $u|_{t=0} = f$.*

The solution $u(t,x)$ is given by

$$u(t,x) = -2[K_x(t;x,x) + K_z(t;x,x)],$$

where $K(t;x,z)$ is obtained as follows:

- *Obtain the scattering data $\{\{-\kappa_n^2, c_n(0)\}_{n=1}^N, b(0;k)\}$ for the Sturm–Liouville problem*

  $$\psi_{xx} + (\lambda - u(0))\psi = 0.$$

  The time evolution of the scattering data is then

  $$\kappa_n = const, \quad c_n(t) = c_n(0)e^{4\kappa_n^3 t}, \quad b(t;k) = b(0;k)e^{8ik^3 t}.$$

- *Define the function*

  $$\begin{aligned} F(t;y) &= \sum_{n=1}^N c_n^2(t) e^{-\kappa_n y} + \tfrac{1}{2\pi}\int_{\mathbb{R}} b(t;k)e^{iky}\,dk \\ &= \sum_{n=1}^N c_n^2(0) e^{8\kappa_n^3 t - \kappa_n y} + \tfrac{1}{2\pi}\int_{\mathbb{R}} b(0;k)e^{8ik^3 t + iky}\,dk \end{aligned}$$

  and solve the Marchenko equation

  $$K(t;x,z) + F(t;x+z) + \int_x^\infty K(t;x,y)F(t;y+z)\,dy = 0$$

  for $K = K(t;x,z)$.

12. Examples

To finish this section, let us work out a few examples of the inverse scattering method outlined above.

Example 27. *Let $u(0,x) = -2\operatorname{sech}^2 x$ (the soliton with $c = 4$), and let $u(t,x)$ be the corresponding solution to KdV.*

This data corresponds to Example 26 above with $U_0 = 2 = N(N+1)$ with $N = 1$.

We found that in this case, $b(0;k) \equiv 0$, which (by the evolution of the scattering data derived above) implies $b(t;k) \equiv 0$ for all $t \geq 0$.

For the discrete spectrum, we worked out that with $N = 1$ we obtain a single eigenvalue $\lambda_1 = -1$ (so $\kappa_1 = 1$), with eigenfunction $\psi_1(x) \propto -\operatorname{sech}(x)$. In particular, because $\int \operatorname{sech}^2(x)\,dx = 2$, the norming constant at $t = 0$ is $c_1(0) = 2^{-\frac{1}{2}}$. The evolution is then given by

$$c_1(t) = 2^{-\frac{1}{2}} e^{4t},$$

and the function $F = F(t,y)$ is

$$F(t,y) = 2e^{8t-y}.$$

The Marchenko equation therefore reduces to

$$K(t,x,z) + 2e^{8t-(x+z)} + 2\int_x^\infty K(t,x,y)e^{8t-(y+z)}\,dy = 0.$$

If we multiply this equation by e^z, we derive that $e^z K(t,x,z)$ depends only on (t,x), so that

$$K(t,x,z) = e^{-z}L(t,x) \quad \textit{for some} \quad L.$$

Inserting this into the equation (after multiplying through by e^z) leads to

$$L(t,x) + 2e^{8t-x} + 2e^{8t}L(t,x)\int_x^\infty e^{-2y}\,dy = 0.$$

Computing the integral leads to

$$L(t,x) = -2\frac{e^{8t-x}}{1+e^{8t-2x}}.$$

Since $K(t,x,z) = e^{-z}L(t,x)$, we do some direct calculations to obtain

$$\begin{aligned} u(t,x) &= -2[K_x(t,x,x) + K_z(t,x,x)] \\ &= -2e^{-x}[L_x(t,x) - L(t,x)] \\ &= -8e^{-x}\frac{e^{8t-x}}{(1+e^{8t-2x})^2} \\ &= -2\operatorname{sech}^2(x-4t). \end{aligned}$$

This is, of course, the soliton with $c = 4$. Cool!

Example 28. *If one chooses*

$$u(0,x) = -N(N+1)\operatorname{sech}^2 x,$$

then (as in Example 26) one obtains N discrete eigenvalues and $b \equiv 0$. In this case, calculations in the spirit of the previous example (but a bit more involved, as we *obtain a linear system rather than a single integral equation) yields the N-soliton solution, which has the asymptotic form*

$$u(t,x) \approx -2 \sum_{n=1}^{N} n^2 \operatorname{sech}^2(n(x - 4n^2 t) \mp x_n) \quad \textit{as} \quad t \to \infty,$$

where x_n are explicit phases. In particular, u asymptotically decouples into N separated solitons.

Further examples in which $b(k)$ is not identically zero are worked out in [1]. The interested reader can find further details there.

References

[1] P.G. Drazin and R.S. Johnson, *Solitons: an introduction.*
[2] L. Evans, *Partial Differential Equations.*
[3] A. V. Gurevich and L. P. Pitaevskii, *Nonstationary structure of a collisionless shock wave.*
[4] C. Kenig, G. Ponce, and L. Vega, CPAM 1993.
[5] R. Killip and M. Visan, *KdV is well-posed in* H^{-1}.
[6] T. Sideris, *Ordinary Differential Equations.*
[7] T. Tao, *Why are solitons stable?*
[8] G. B. Whitham, *Linear and Nonlinear Waves.*
[9] G. B. Whitham, *Non-linear dispersive waves* (1965).

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