Classification of simple amenable C^*-algebras of stable rank one

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$\star$ $M_n(\mathbb{C})$, the $n \times n$ matrices, is a unital simple finite dimensional C^*-algebra. Every finite dimensional C^*-algebra is a finite direct sum of matrix algebras.

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$$\begin{aligned} \langle U(f), U(g) \rangle &= \int_X \overline{U(g)} U(f) d\mu \\ &= \int_X \bar{g} f d\mu = \langle f, g \rangle \end{aligned}$$

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for all $f, g \in C(X)$ and

$$(U\pi(a)U^*)(f)(\xi) = \pi(a \circ \sigma^{-1})(f(\xi))$$

for all $a \in C(X)$, $f \in L^2(X, \mu)$ and $\xi \in X$.

The C^*-algebra generated by $\pi(C(X))$ and U is denoted by $C(X) \rtimes_{\sigma_\theta} \mathbb{Z}$ (the crossed product of $C(X)$ by $\mathbb{Z}$ with σ). It is a simple C^*-algebra.

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Is $C(\mathbb{T}) \rtimes_{\sigma_\theta} \mathbb{Z}$ isomorphic to A_θ?

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Let A be a unital C^*-algebra. $u \in A$ is a unitary if $u^*u = uu^* = 1$.
$U(A)$: the group of unitaries. $U(A)_0$ the connected component containing 1_A.

Let $j_n : U(M_n(A)) \to U(M_{n+1}(A))$ be $j_n(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$.

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$$K_1(A) = \lim_{n\to\infty} (U(M_n(A))/U(M_n(A))_0, \bar{j}_n).$$

Example

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- ⋆ $$K_0(C(\mathbb{T}) \rtimes_{\sigma_\theta} \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}_\theta \text{ and } K_1(C(\mathbb{T}) \rtimes_{\sigma_\theta} \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}.$$

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Let A and B be two unital AF-C^*-algebras. Then $A \cong B$ iff

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$A\mathbb{T}$-algebras: inductive limits of finite direct sums of $C(\mathbb{T}) \otimes M_k$

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It was later proved by Dadarlat and Gong that the condition of no dimension growth can be replaced by **slow dimension growth**.

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if and only if for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, there exists a finite dimensional C^*-algebra B ($B = \oplus_{i=1}^{k} M_{R(i)}$) such that

$$\mathrm{dist}(x, B) < \epsilon$$

for all $x \in \mathcal{F}$.

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(i) $\|px - xp\| < \epsilon$ for $x \in \mathcal{F}$,
(ii) $pxp \in_\epsilon C$ for $x \in \mathcal{F}$ and
(iii) $1 - p$ is equivalent to a projection in $\overline{aAa}$.

$$\left(\begin{array}{c|c} \text{qAq} & \\ \hline & \\ & \text{B} \\ & \end{array}\right)$$

where $q = (1 - p)$ and $\dim B < \infty$.

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(reference: Ann. of Math., **157** (2003), 521-544; Duke Math. J., **125** (2004), 91-119)

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$\star$ *Every unital separable amenable simple C^*-algebra with tracial rank zero satisfying the UCT is a unital simple AH-algebra (with no dimension growth and with real rank zero).*

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- Let X be a connected compact metric space,
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$$\prod_{k=1}^{m} e^{ia_k} = v.$$

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Let X be a connected CW complex and let $\alpha : X \to X$ be a minimal homeomorphism. Suppose that α is unique ergodic.

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if and only if the associated rotation number of α has irrational values.

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Every irrational rotation algebra is an $A\mathbb{T}$-algebra with real rank zero.

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Let X be a compact metric space with finite covering dimension, let $\alpha, \beta : X \to X$ be two minimal homeomorphisms and let $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ and $A_\beta = C(X) \rtimes_\beta \mathbb{Z}$.

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In the above definition, if C can be chosen to be a finite dimensional C^*-subalgebra then
$TR(A) = 0$. If $TR(A) \leq 1$ but $TR(A) \neq 0$ then we will write $TR(A) = 1$.

Theorem

(G. Gong) *Every unital simple AH-algebra with no dimension growth has tracial rank no more than one.*

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In fact, every unital simple AH-algebra with (very) slow dimension growth has tracial rank no more than one.

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If we know $TR(A) \leq 1$, when $TR(A) = 0$?

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- $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$
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Let A and B be two unital separable simple amenable C^*-algebras with $TR(A) \leq 1$ and $TR(B) \leq 1$ which satisfy the UCT.

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Moreover, $\mathcal{Z}$ has no non-trivial projection, has stable rank one and is an inductive limit of type I C^*-algebras.

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(2) $A \cong A \otimes \mathcal{Z}$;
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(4) A is isomorphic to a unital simple AH-algebra with has no dimension growth.