

RAY AND WAVE DYNAMICS IN THREE DIMENSIONAL
ASYMMETRIC OPTICAL RESONATORS

by

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Whispering gallery modes formed by total internal reflection in dielectric optical resonators, such as disks and spheres, can have extremely high Q-factor and small mode volume, leading to their application in cavity QED and microlasers. Isotropic emission from these modes, however, hinders widespread applications. Highly asymmetric but convex dielectric resonators with two degrees of freedom have exhibited directional emission, but with relatively low Q-factor. These resonators also provide a model system for investigating quantum/wave chaos in a system with two degrees of freedom.

This dissertation presents experimental and numerical studies of deformed fused-silica microspheres. Techniques are developed to fabricate deformed microspheres with no axial symmetry. In addition to providing directional emission from high-Q modes, deformed microspheres present an opportunity to experimentally investigate the

dynamics of a mixed phase space system with three degrees of freedom where effects such as Arnold diffusion can occur. Optical properties such as emission directionality, Q-factor, and resonance spectrum are investigated for microspheres over a range of deformations and input conditions. Analysis of the results emphasizes the subtle interplay between ray and wave effects in determining emission directionality and resonance lifetime.

Directional emission is observed in microspheres with surprisingly small deformation ($\approx 1\%$). The four lobed far-field emission pattern, which has not previously been observed or predicted, is explained as tunneling escape that is directed by non-perturbative features in the classical phase space. In strongly deformed microspheres ($> 5\%$) this emission mechanism is found to be suppressed by rapid refractive escape with directionality that is well explained by dynamical eclipsing in a 2D model. The 2D model fails to explain the Q-factor for the excited modes in both deformation regimes. A full three dimensional ray model is presented that illuminates the role of Arnold diffusion, which allows a ray to transition from glancing incidence to refractive escape and does not occur in two dimensions.

Increased understanding of the dynamics of asymmetric resonators has also led to practical advances. As an example, free-space coupling to modes with $Q > 10^7$ is demonstrated. The practical implementation of free-space coupling is facilitated by dynamical eclipsing.

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May God bless you all.

DEDICATION

For my Dad

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CHAPTER I

INTRODUCTION

During the past fifteen years dielectric optical resonators have attracted great interest for both commercial and research applications¹. In highly symmetric resonators, such as disks and spheres, light circulates with extremely low loss around the curved interface at nearly glancing incidence. In fused-silica microspheres these whispering gallery (WG) modes can provide remarkably long photon storage times $\tau > 4 \mu\text{s}$ ($Q \equiv \tau \cdot \omega > 8 \times 10^9$)^{2,3} within a very small resonator ($D \approx 100 \mu\text{m}$). The unique properties of glass microspheres make them an ideal basis for a variety of low threshold microlaser devices. Recently a number of novel lasers based on silica microspheres have been demonstrated. In one case lasing was achieved by depositing an Er^{3+} doped solgel layer on the outside of the microsphere⁴. Another was a Raman laser with a pump threshold of only $84 \mu\text{W}$ and a quantum efficiency of 36%⁵. In addition, lasing thresholds as low as 200 nW have been observed in Nd doped silica microspheres⁶. The high Q-factor and small effective mode volume of dielectric cavities also make them extremely well suited to cavity quantum electrodynamics (cavity QED) experiments⁷. Enhanced spontaneous emission has been observed from semiconductor nanocrystals embedded in polystyrene spheres⁸. A high-Q silica microsphere/nanocrystal system has also been demonstrated which may lead to realization of the strong-coupling regime of cavity QED for atoms and

semiconductors⁹. Fused-silica microspheres have also been developed for commercial applications such as an optical communications add/drop filter¹⁰ and a sensor for detection of trace-gas¹¹ and biological molecules¹². A recently developed high-Q silica-on-silicon-chip toroid-shaped resonator¹³ promises to greatly increase such commercial applications.

Despite their remarkable properties and demonstrated successes, many applications of dielectric resonators suffer from the inherent isotropic emission that results from their high symmetry. A common solution to this problem is to provide input/output coupling through a near-field coupling device. For glass microspheres this can be done using a high index prism, slab waveguide, or tapered optical fiber¹⁴. Each of these coupling mechanisms significantly degrades the high intrinsic Q-factor of the microsphere when placed in contact with it. High Q-factors can be maintained, but only at the expense of sensitive, and typically unstable, positioning of the microsphere and coupler separated by a small gap.

In 1994 it was predicted that a dielectric resonator with asymmetric, but everywhere convex, cross section could provide highly directional emission while maintaining a relatively large Q-factor¹⁵. A ray optics model, treating light in the asymmetric cavity as a particle bouncing off of a curved boundary, indicated that the cavity Q would remain unspoiled up to a threshold deformation, at which point the emission would become directional and the Q would decrease rapidly with increasing deformation. The genesis of this interesting, non-perturbative behavior is the chaotic dynamics present in the asymmetric billiard model. An interesting puzzle arises when

one considers how the chaotic dynamics of the ray optics model can be manifest in the optical properties of a resonator. The behavior of the resonator should ultimately be determined by solution of Maxwell's equations which, however, do not display chaos.

This situation is closely analogous to the problem faced when trying to understand the behavior of a quantum mechanical system for which the corresponding classical dynamics are chaotic. This issue is currently under active investigation in a wide variety of physical systems¹⁶. One of the most well studied systems regarding quantum chaos is the hydrogen atom in a strong magnetic field¹⁷. A field of research that is closely related to the study of asymmetric optical resonators is ballistic electron transport in high purity semiconductor nanostructures¹⁸. More recently, advancements in atom cooling techniques have led to the demonstration of ultra-cold atoms confined to an "optical billiard" produced by a rapidly scanned laser beam¹⁹.

Asymmetric resonant cavities (ARCs) are particularly attractive systems for the study of wave chaos effects. One reason is that in contrast to electron billiards, optical materials such as fused-silica are readily available which provide extremely low absorption and scattering. This allows one to study dynamics over much longer time scales and therefore investigate the dynamics of systems which are only weakly chaotic. Also, optical billiards closely approximate the ideal hard walled billiard whereas the confining potentials in electron and atom billiards produce soft walls and therefore more complicated dynamics. Furthermore, optical billiards do not require one or more holes in the billiard wall. Rather than a particle escaping when it arrives at a certain location on the billiard wall, light escapes from a dielectric optical cavity when its angle of incidence

exceeds the critical angle for total internal reflection. All of these features of optical billiards allow one to study wave chaos in a regime sometimes referred to as soft chaos. Soft chaos describes a system where not all trajectories are chaotic, but rather a mixed phase space exists where regular periodic and quasiperiodic trajectories persist and are densely intermingled with chaotic trajectories. These soft chaos systems are very interesting because they represent a regime where perturbation theory yields valuable results and yet is not globally applicable. On the other hand, traditional techniques for analyzing quantum chaotic systems, such as random-matrix theory, are limited to systems that exhibit hard chaos. As such, optical billiards are an excellent system for studying the subtle interplay between quantum and classical dynamics in the transitional regime between regularity and chaos.

Asymmetric dielectric resonators have been fashioned from a variety of different media. The earliest ARCs studied were liquid droplets whose shape oscillated between oblate and prolate while falling²⁰. When the ethanol droplets containing laser dye were pumped, enhancement and suppression of lasing emission was observed with characteristic patterns as the shape of the droplet changed. More recently lasing in pendant droplets deformed by an applied electric field has been studied²¹. Cylindrical ARCs have also been realized in a variety of experiments employing liquid jets. Non-circular liquid jets can be formed by forcing a stream through a rectangular orifice²² or by distorting a circular jet with air flow. Yet another type of ARC is the semiconductor microdisk laser which is etched to have an oval shape. Such asymmetric microdisk lasers

supporting “bow-tie” modes have demonstrated great promise as high power microlasers²³.

In this dissertation we present experimental and numerical studies of deformed fused-silica microspheres, which represent a new type of asymmetric dielectric optical resonator. The deformed microspheres we fabricate are studied as passive cavities with no active gain medium. This is in contrast to almost all previous studies on ARCs which have been done with lasing systems. The excellent optical properties of fused-silica and the strong mode confinement in microspheres allow for the largest unspoiled Q-factor of any dielectric resonator. Deformed microspheres also provide the unique advantage of being a solid-state asymmetric cavity where the deformation may be adjusted to study a single resonator across a range of deformations.

More importantly, deformed microspheres are the first asymmetric WG resonator to be studied, for which the ray dynamics are not reducible to a two dimensional billiard system. As opposed to the oscillating droplets which retain rotational symmetry about the vertical axis, we have developed techniques to fabricate deformed microspheres which do not possess any rotational symmetry. The fully three dimensional nature of deformed microspheres is significant because a qualitative difference exists between the classical dynamics of systems with two versus three degrees of freedom. In partially chaotic systems with two degrees of freedom, chaotic trajectories are rigidly confined in phase space by the remaining non-chaotic trajectories. For systems with more than two degrees of freedom this confinement is not complete and a process called Arnold diffusion allows any chaotic trajectory to eventually explore every finite region of the

phase space. Arnold diffusion is a phenomenon which has proven to be particularly challenging from a theoretical standpoint and has not been investigated to the extent of some other aspects of chaotic dynamics. The role of Arnold diffusion in a quantum or wave system is not well understood. It is hoped that experimental measurements on systems with more than two degrees of freedom might motivate increased attention to this interesting subject.

We have observed directional emission from deformed microspheres with surprisingly weak ($\approx 1\%$) deformations. The phase space of these resonators is dominated by non-chaotic trajectories, which in a 2D ray optics model would suggest that no directional emission should be observed. We interpret the emission as arising due to the breaking of rotational symmetry of the deformed microsphere. The emission pattern observed from the weakly deformed microspheres is different than what has previously been observed in other asymmetric dielectric cavities. Detailed experimental and theoretical studies show that the emission represents directional tunneling escape from the resonator. The directionality is determined by non-perturbative phase space structures in a way that is similar to an effect known in ARCs as dynamical eclipsing. This unique emission pattern is a sensitive probe of the relative importance of wave and ray effects in the cavity. It can also be exploited to achieve practical free-space coupling to high-Q WG modes of deformed fused-silica microspheres.

This dissertation will begin with an introduction to the theory of whispering gallery modes in Chapter II. In addition to the solution of the wave equation in a spherical dielectric, further physical insight into the nature of high-Q WG modes will be

provided by examining the effective radial potential as an analogous one dimensional quantum well. Chapter III will review the necessary basics of Hamiltonian dynamics of chaotic systems. The emphasis will be on the theorems and techniques that will be found particularly useful in understanding the mixed phase space of near-integrable billiard systems. In Chapter IV we will describe the techniques that we have developed for the fabrication of fused-silica microspheres and deformed microspheres. The experimental techniques of launching and characterizing WG modes as well as measuring directional emission from deformed microspheres will also be discussed. Chapter V will focus on the comparison of experimental results obtained from deformed microspheres with theoretical ray tracing models. A two dimensional ray optics model will be used to explain the directional emission pattern observed for strongly deformed microspheres. It will also be shown that the 2D model fails to explain the observed WG mode lifetimes. A 3D ray tracing model is developed to demonstrate the crucial influence of Arnold diffusion on the orbital dynamics of light in a deformed microsphere²⁴. In Chapter VI we will discuss the influence of wave effects on the optical properties of deformed microspheres. Experimental evidence of a new directional tunneling emission mechanism in weakly deformed microspheres is presented²⁵. The impact of wave effects on phase space transport in a three dimensional cavity are discussed. Chapter VII demonstrates free-space coupling to high-Q modes of a deformed glass microsphere. This is intended as an example of the practical applications that are made possible with deformed glass microspheres. In addition, the investigation of free-space coupling reveals further interesting examples of the rich interplay between ray and wave effects in

a chaotic microcavity. Finally, Chapter VII provides a summary of the experimental and numerical results as well as suggestions for the directions that future research in this area might take.

CHAPTER II

THEORY OF WHISPERING GALLERY MODES IN A SPHERICAL RESONATOR

Dielectric microcavities, such as fused silica microspheres, have attracted interest^{1,26} because they support high-Q resonances called whispering-gallery (WG) modes. These resonances are named for their resemblance to the acoustic modes of the whispering gallery of St. Paul's Cathedral. Lord Rayleigh explained that acoustic waves travel with minimal attenuation along the curved walls of the cathedral by repeated glancing reflection allowing two people positioned near the wall at opposite sides of the dome to communicate through whispers while a listener in the center hears nothing²⁷. Optical WG modes were first observed in scattering experiments by G. Mie²⁸ and WG modes are therefore often referred to as "Mie resonances". Another common term for these modes is "Morphology Dependent Resonances" (MDRs) due to their sensitive shape dependence.

In a dielectric cavity the repeated reflections will have almost no attenuation when the angle of incidence χ , measured from the surface normal, is greater than the critical angle for total internal reflection. When a ray is transmitted across an interface between two materials with different refractive indices the propagation direction of the ray is changed. Refraction is governed by Snell's law,

$$n_1 \sin \chi_i = n_2 \sin \chi_r \quad (1.1)$$

where χ_i is the angle of incidence in the material with index n_1 and χ_r is the angle of refraction in the material with index n_2 . If $n_1 > n_2$ Snell's law does not provide real values for χ_r when χ_i is greater than a certain angle. This critical angle χ_c is the angle of incidence for which a ray cannot be transmitted from a high index material into a low index material. When a dielectric cavity with index of refraction n is surrounded by air the critical angle is given by

$$\sin \chi_c \approx \frac{1}{n} \quad (1.2)$$

since the index of refraction for air is approximately one.

In a circular dielectric cavity the angle of incidence for each successive reflection will remain the same due to symmetry. Therefore, rays starting with $\chi > \chi_c$ will be permanently trapped inside the dielectric. When a set of rays is launched with $\chi > \chi_c$, at each reflection the curved surface of the sphere refocuses the beam. If the beam makes an integral number of reflections and returns to its starting position in phase, constructive interference will occur. Such a set of rays, trapped in the cavity by total internal reflection, corresponds to a long lived WG mode.

In the limit of plane waves incident on a flat interface, total internal reflection is perfect and there is no energy transmitted across the interface. There is a transmitted wave vector, however, which has a real component parallel to the interface and an imaginary component perpendicular to the surface. This is called an evanescent wave and the

imaginary wave vector perpendicular to the interface indicates the exponential decay of the wave amplitude with distance from the interface. When total internal reflection occurs with non-plane waves at a curved interface there is a real component of the transmitted wave vector which is tangential to the surface and does allow a small amount of energy transfer across the curved interface. Thus WG modes will radiate weakly and have a finite lifetime.

One often desires to couple light in and out of a WG mode efficiently. The symmetry of Snell's law indicates that the input coupling efficiency by direct illumination will be just as small as the emission rate. Thus direct illumination is clearly not an efficient means of exciting long lived WG modes. An effect analogous to quantum mechanical tunneling, however, can be used to couple light into a WG mode. Frustrated total internal reflection can occur when two high index materials are separated by a small layer of lower index. If light in the first material reflects with an angle of incidence greater than the critical angle an evanescent wave will be produced in the low index material in the gap. Inside the second high index material the imaginary component of the transmitted wave vector becomes real again. Thus if the exponential tail of the evanescent wave extends into the second high index material a portion of the incident beam will be transmitted across the gap into the second material as shown in figure 1. The amplitude of the transmitted wave is determined by the amplitude of the evanescent wave at the second interface and is therefore exponentially dependant on the size of the gap between the two high index materials.

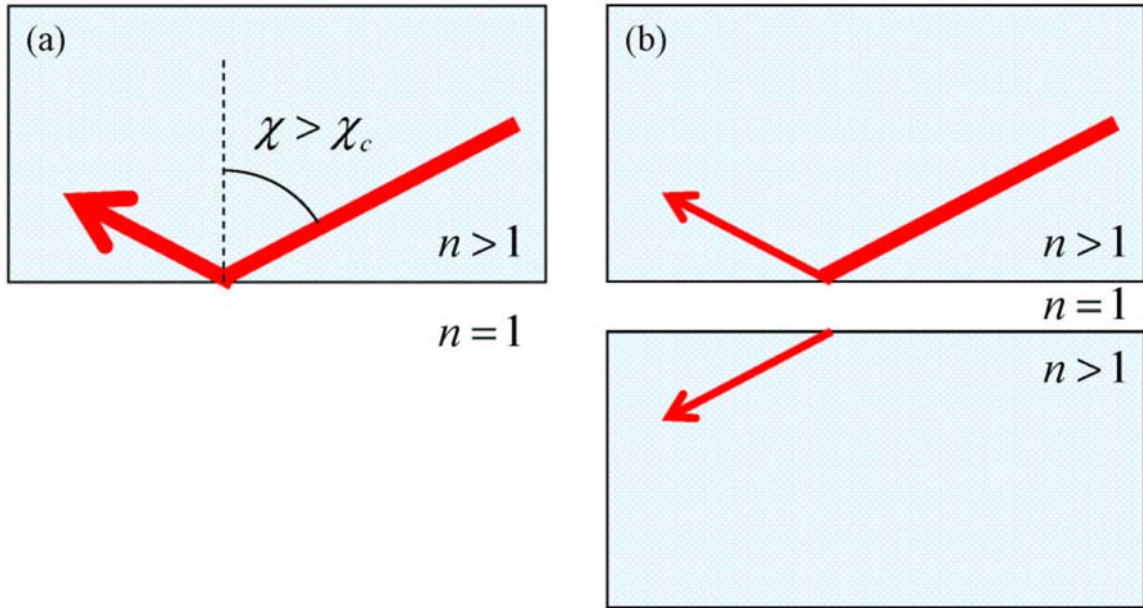


FIGURE 1. Total internal reflection occurs when light reflects off the interface between a high index material and a low index material with an angle of incidence greater than the critical angle $\chi_c = \sin^{-1}(1/n)$ (a). Frustrated total internal reflection is partial transmission of the incident light into a second high index material across a low index gap (b). The amplitude of the transmitted beam is exponentially dependant on the gap size.

Solution of Wave Equation

Although some properties of whispering gallery modes are well suited to a geometrical optics treatment, a complete understanding requires solution of Maxwell's equations. The Helmholtz wave equation for electromagnetic radiation in a homogeneous dielectric of index n can easily be derived from Maxwell's equations and is typically

written as

$$\nabla^2 \mathbf{E} + n^2 k^2 \mathbf{E} = 0 \quad (1.3)$$

where the wave number in vacuum, k , is given by ω/c . It has been shown^{29, 30} that the vector solution of a highly symmetric system can be conveniently obtained by solution of the scalar wave equation

$$\nabla^2 \psi + n^2 k^2 \psi = 0. \quad (1.4)$$

For spherical symmetry the transverse electric (TE) and transverse magnetic (TM) modes, with zero radial electric and magnetic field components respectively, are decoupled. The electric field for each case is obtained from the scalar potential ψ by

$$\begin{aligned} \text{TE modes} \quad \mathbf{E} &= \nabla \times (\mathbf{r} \psi) \\ \text{TM modes} \quad \mathbf{E} &= \nabla \times \nabla \times (\mathbf{r} \psi). \end{aligned} \quad (1.5)$$

In spherical coordinates equation (1.4) is separable and the solutions have the form

$$\psi_{lm}(r, \theta, \phi) = z_l(nkr) Y_{lm}(\theta, \phi) \quad (1.6)$$

where $z_l(nkr)$ may be any spherical Bessel function, $Y_{lm}(\theta, \phi)$ are the spherical harmonics. We are interested in the solution for a spherical dielectric of radius a and index n in air,

$$\psi(r, \theta, \phi) = \begin{cases} \sum_{l,m} a_{lm} j_l(nkr) Y_{lm}(\theta, \phi), & r \leq a \\ \sum_{l,m} b_{lm} h_l^{(1)}(kr) Y_{lm}(\theta, \phi), & r > a \end{cases} \quad (1.7)$$

where $j_l(x)$ is the spherical Bessel function and $h_l^{(1)}(x)$ is the spherical Hankel function of the first kind. The continuity of ψ across the sphere surface requires

$$\frac{b_{lm}}{a_{lm}} = \frac{j_l(nx)}{h_l^{(1)}(kx)} \quad (1.8)$$

where $x \equiv ka$ is the size parameter. The requirement that $d\psi/dr$ must also be continuous across the sphere surface leads to the condition

$$f_{lm}(x) = nj_{l-1}(nx) - n(l+1)j_{l+1}(nx) - \frac{j_l(nx)}{h_l^{(1)}(x)} [lh_{l-1}^{(1)}(x) - (l+1)h_{l+1}^{(1)}(x)] = 0. \quad (1.9)$$

The roots of $f_{lm}(x)$ are the resonant size parameters where this condition is met. The roots are complex in general, and the imaginary component of k describes to the exponential decay of the quasibound states due to the open nature of the resonator. The eigenmodes of the system are labeled by the set of mode indices μ , l , m , and ν , where μ indicates TE or TM polarization and ν indicates the mode associated with the ν th root of $f_{lm}(x)$. Aside from the polarization description necessary in optics, these mode indices are analogous to the quantum numbers which describe the bound states of the hydrogen atom. An alternate notation is often used in the literature on dielectric microspheres where l , m , and ν , are replaced by n , m , and l , and the index of refraction is also m . In our notation the mode number, l , corresponds to the total angular momentum of the field. The azimuthal mode number, m , can take on the values $m = -l \dots 0 \dots +l$ and corresponds to the z-component of the angular momentum. The order number, ν , is equal to the number of radial maxima of the electric field.

Contour plots representing the field intensity for various TE modes in the $x-y$ and $x-z$ planes are shown in figures 2 and 3. These plots demonstrate the dependence

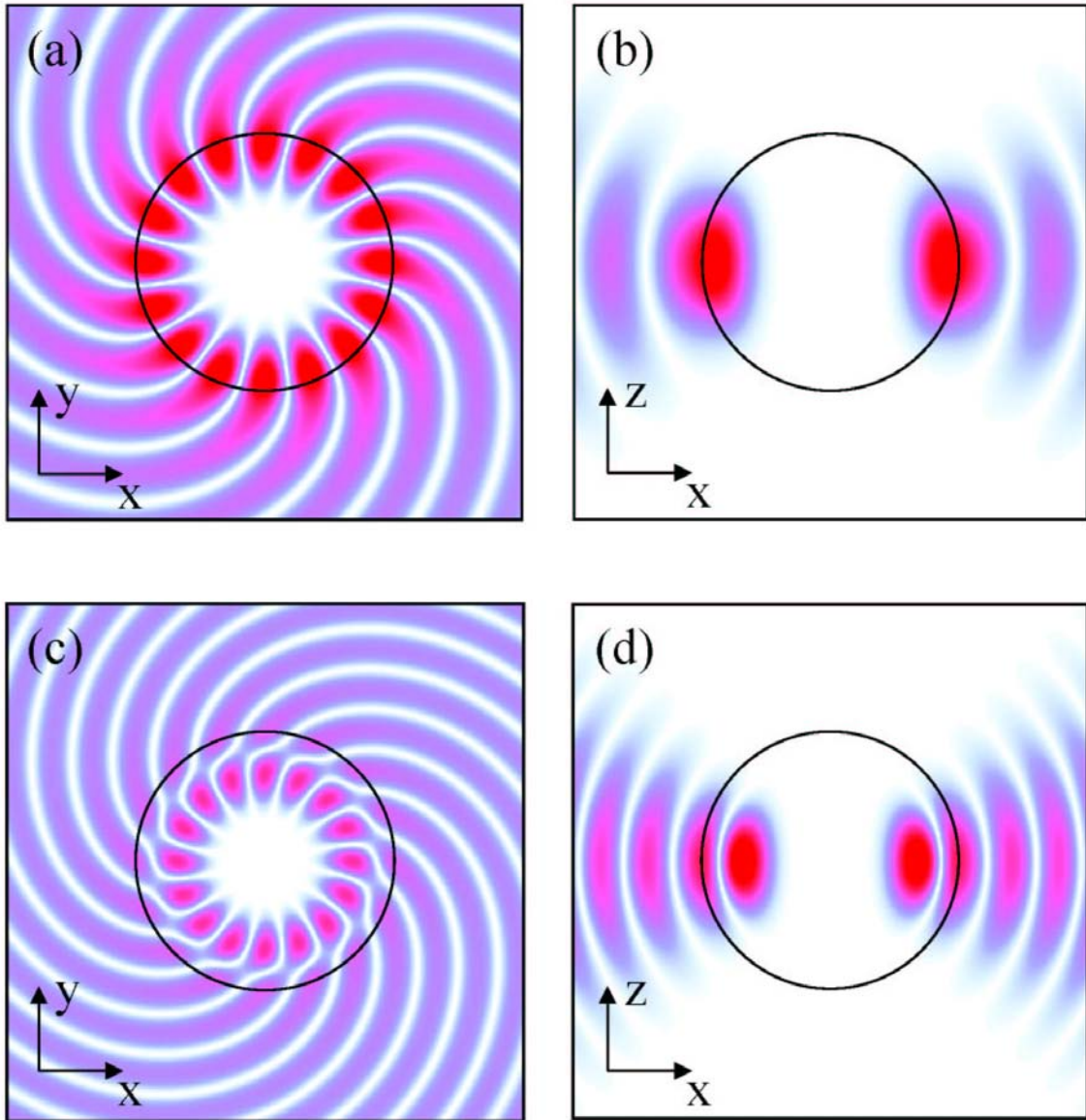


FIGURE 2. False color intensity plots of quasinormal modes in a dielectric sphere. The modes shown are $l=8, m=8, \nu=1$ (a, b) and $l=8, m=8, \nu=2$ (c, d).

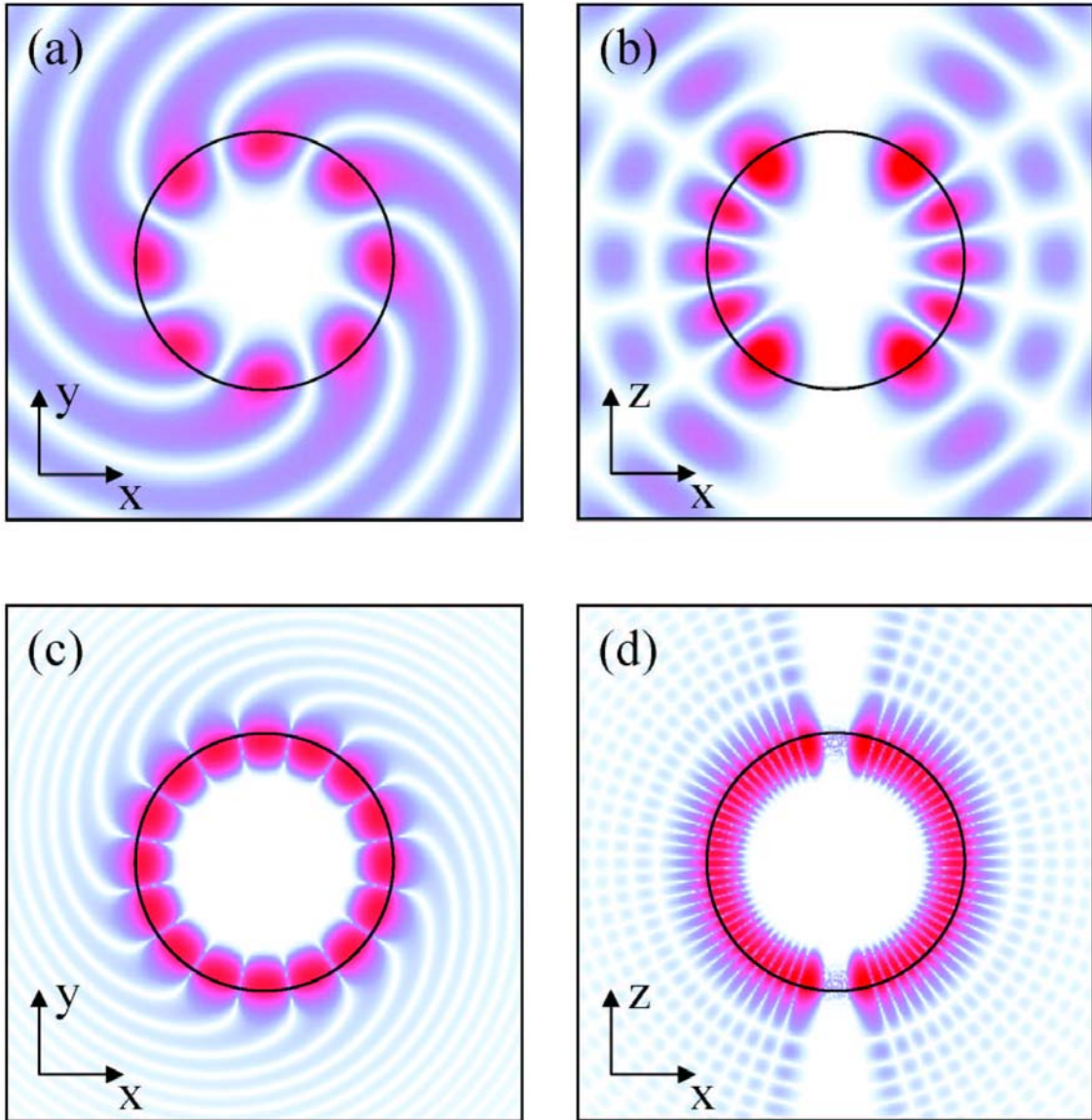


FIGURE 3. False color intensity plots of quasinormal modes in a dielectric sphere. The modes shown are $l=8$, $m=4$, $\nu=1$ (a, b) and $l=40$, $m=8$, $\nu=1$ (c, d).

of the spatial mode pattern on the three mode numbers. For each mode there are $2m$ nodes around the circumference of the sphere and $l - |m| + 1$ lobes in the range $0 < \theta < \pi$. These lobes are centered at the equator ($\theta = \pi/2$) and extend to $\theta_{\max} \approx \pi/2 \pm \cos^{-1}(m/l)^{31}$. As mentioned above, ν determines the number of radial maxima in the field and modes with larger ν extend further into the exterior of the sphere. Each of the modes shown corresponds to a single $m > 0$ mode orbiting counterclockwise within the sphere. Standing waves are obtained by superposition of $\pm m$ modes.

Effective Radial Potential

Additional physical insight into WG modes can be gained by comparing the dielectric cavity to the analogous finite square well in quantum mechanics³². Equation (1.4) can be rewritten to resemble the Schrödinger equation as,

$$-\nabla^2 \psi + k^2 (1 - n^2) \psi = k^2 \psi. \quad (1.10)$$

In this form the quantity $k^2 (1 - n^2)$ takes on the role of an attractive potential and k^2 corresponds to the energy of a state. For a circular dielectric cavity of radius, a , surrounded by air the index of refraction is,

$$n(r) = \begin{cases} n, & (r < a) \\ 1, & (r > a) \end{cases}. \quad (1.11)$$

The circular symmetry of such a cavity allows one to perform a separation of variables and write the radial equation,

$$-\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right] S(r) + V'(r) S(r) = k^2 S(r), \quad (1.12)$$

where the effective potential,

$$V'(r) = k^2 (1 - n^2(r)) + \frac{l^2}{r^2} \quad (1.13)$$

includes a so-called centrifugal barrier resulting from the conservation of angular momentum, l . Although this effective potential is complicated by the fact that it depends explicitly on the wave vector it is possible to deduce a great deal of information about WG modes from it. Examining the effective potential plotted in figure 4 we can see that quasibound states will exist for a specific range of 'energy levels,' k^2 , with limits corresponding to the bottom of the well and the top cusp of the barrier. The largest wave vector that corresponds to a quasibound state, therefore, is $k_{\max} = l/a$. The lowest energy quasibound state has energy, $k^2 = k^2(1 - n^2) + l^2/a^2$, and wave vector, $k_{\min} = l/na$. It is helpful to relate the wave vector of an energy state to its impact parameter, $b = l/k$, which describes the perpendicular distance between an incoming ray and a parallel line emerging radially from the sphere center. The range of wave vectors which represent quasibound states, $l/a > k > l/na$, corresponds to a range of impact parameters, $a < b < na$ which will produce trapped rays by tunneling into the cavity. A ray with $k = l/b$ where $a < b < na$ will have classical turning points at either $r = b$ corresponding to the closest approach of the ray outside the cavity, or $r = b/n$ and $r = a$ corresponding respectively to the deepest penetration of the ray inside the cavity and repeated reflection

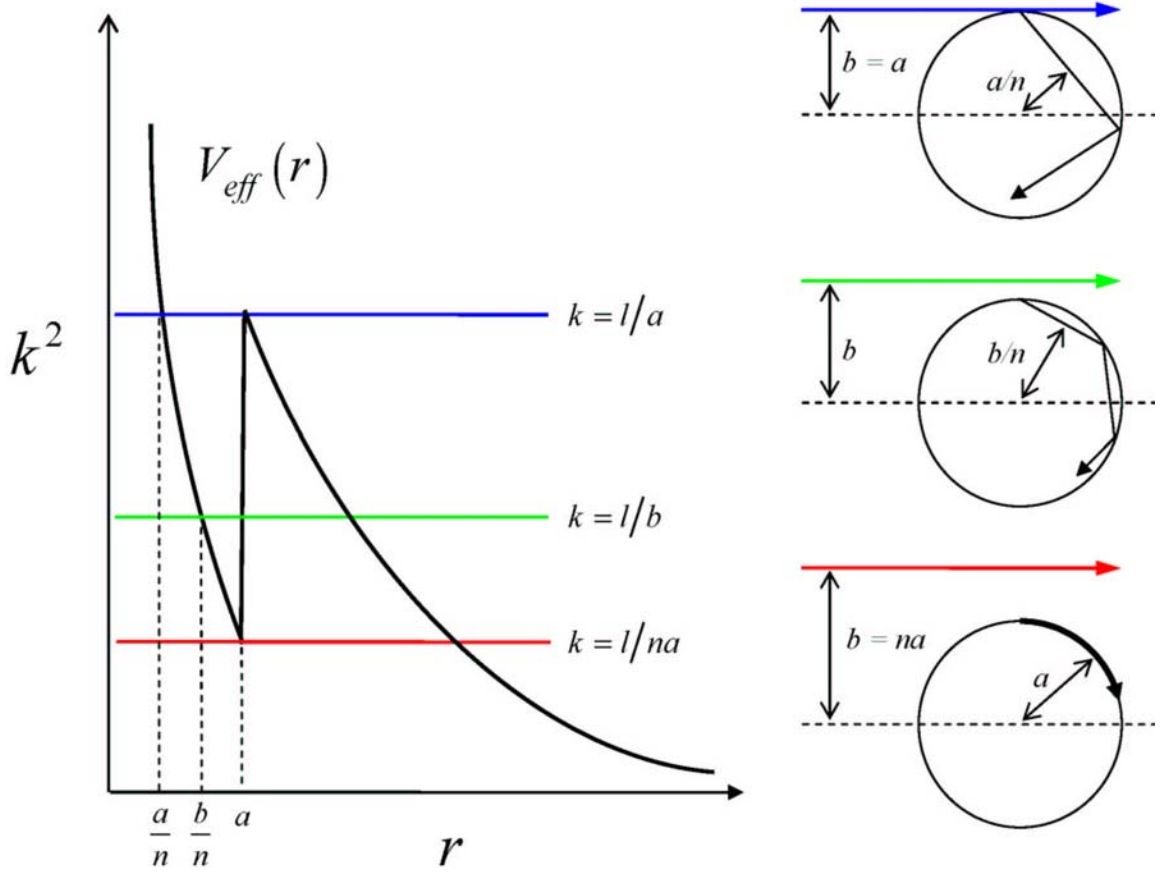


FIGURE 4. The effective radial potential is plotted. Three cases leading to WG modes are indicated. In blue, a ray tangent to the sphere is refracted and undergoes repeated reflection at the critical angle. In red, a ray tunnels into a maximally confined mode with $\sin \chi = 1$. A general case with impact parameter between these extremes is shown in green. It should be noted that the dependence of the well depth on k has been suppressed in this figure.

off the surface with $\sin \chi = b/na$. In the limit where $k = l/na$ the bound state is at the bottom of the potential well and both turning points inside the well are at $r = a$ implying a circular trajectory or $\sin \chi = 1$. When $k = l/a$, on the other hand, $\sin \chi = \sin \chi_c = 1/n$ and the state is just barely bound by total internal reflection. In fact we can see that *total*

internal reflection is never actually *total* in a dielectric cavity due to evanescent leakage to the outside by tunneling through the finite barrier. Despite this, the tunneling rate can become extremely small for states with the smallest wave vectors. As the wave vector decreases towards $k = l/na$ the barrier becomes increasingly high and wide leading to an increasingly well confined mode.

Features of Whispering Gallery Modes

Whispering gallery modes are most commonly associated with small ν , large l and $m \approx \pm l$. The combination of small ν and large l corresponds to a mode whose field is strongly concentrated near the sphere's surface. The condition $m \approx \pm l$ requires that the field also be localized to a narrow ring around the sphere's equator. Although this second condition is not absolutely necessary for high-Q WG modes it is often desirable as it provides an extremely small effective mode volume and emission concentrated in the equatorial plane. Since the angular momentum mode numbers l and m are large for WG modes they can therefore be related to familiar geometrical quantities. The same is not true for ν , which is generally small for high Q modes.

The largest value that l can take on is approximately

$$l_{\max} \approx nka = \frac{2\pi a}{\lambda/n}. \quad (1.14)$$

This leads to the physical interpretation that the maximum l mode consists of l total internal reflections, each separated by an optical path length of one wavelength inside the

sphere (λ/n). The mode number l can also be related to the angle of incidence inside the sphere as

$$l \approx nka \sin \chi. \quad (1.15)$$

Furthermore, by considering the change in wavelength associated with a difference in l of one we can calculate the free spectral range of a cavity for large l to be,

$$\Delta\lambda_{fsr} = \lambda^2 / 2\pi na \quad (1.16)$$

or alternatively,

$$\Delta\omega_{fsr} = c/na. \quad (1.17)$$

In a perfect sphere, modes with the same ν and l but different m would be degenerate. In fact, since there is no natural z -axis in a sphere, any common input coupling techniques would only excite $|m| \approx l$ type modes, which are confined to a great circle, although they could have arbitrary inclination. In a spheroid with small ellipticity the degeneracy is removed and modes launched in a plane inclined with respect to the equator (defined by the plane perpendicular to the symmetry axis) will precess about the symmetry axis³³.

These precessing modes have $|m| \neq l$ and field distributions like those shown in figure 3.

It has been shown³¹ that modes in an ellipsoid with $e = (r_{\max} - r_{\min})/a$ are red-shifted from the resonant frequency ω_0 of the equal volume sphere of radius a such that

$$\omega(m) = \omega_0 - \omega_0 \frac{e}{6} \left(1 - \frac{3m^2}{l(l+1)} \right). \quad (1.18)$$

This can be differentiated to determine the approximate precession frequency Ω of $|m| \neq l$

modes and the splitting between adjacent m modes

$$\Omega \approx \frac{d\omega}{dm} = \omega_0 |e| \frac{m}{l(l+1)} \approx \frac{\omega_0 |e|}{nka} \cos \theta \quad (1.19)$$

where $\cos \theta = m/l$ and θ is the inclination of the excitation plane relative to the equator.

The last expression assumes a large size parameter such that $l+1 \approx l = l_{\max}$.

One of the most intriguing aspects of whispering gallery resonators is the extremely high quality factor, Q , which is achievable. The Q of a resonance is defined as

$$Q = \frac{2\pi \text{ Stored energy}}{\text{Energy lost per cycle}} = \frac{-\omega_0 U}{dU/dt} \quad (1.20)$$

where U is the energy stored in the resonance and ω_0 is the resonance frequency.

Equation (1.20) can be solved to demonstrate the exponential decay of energy stored in the resonator with a time constant, $\tau = Q/\omega_0$. In addition, it can be shown that the resonance line shape is Lorentzian

$$\frac{1}{(\omega - \omega_0)^2 + (\omega_0/2Q)^2} \quad (1.21)$$

such that Q is inversely proportional to the full width at half-maximum of the resonance

$$\delta\omega = \frac{\omega_0}{Q}. \quad (1.22)$$

The total Q of a WG mode can be considered as resulting from several loss mechanisms and written as

$$Q^{-1} = Q_{int}^{-1} + Q_{s.s.}^{-1} + Q_{cont}^{-1} + Q_{mat}^{-1} + Q_{coup}^{-1} \quad (1.23)$$

where Q_{int}^{-1} represents intrinsic radiative losses determined by geometry, $Q_{s.s.}^{-1}$ represents

losses due to scattering off of surface roughness, Q_{cont}^{-1} represents losses due to contaminants which attach to the surface, Q_{mat}^{-1} represents material losses due to absorption and Rayleigh scattering, and Q_{coup}^{-1} represents losses introduced by input/output coupling mechanisms. Coupling to WG modes is discussed in more detail in the next chapter. For the current discussion it suffices to point out that Q_{coup}^{-1} can be made arbitrarily small at the expense of coupling efficiency. For low ν modes in fused silica microspheres ($\sin \chi \approx 1$) Q_{int}^{-1} results from tunneling and decreases exponentially with increasing size. Therefore, in practice the Q-factors for WG modes in spheres with $a/\lambda \gg 1$ are almost always dominated by other loss mechanisms.

Size does have a strong effect on Q, however, through surface scattering loss. A theoretical estimate of the surface scattering rate in WG modes can be obtained by modeling surface roughness features as Rayleigh scatters^{2,3}. For total internal reflection at glancing incidence this model predicts

$$Q_{s.s.} = \frac{3n^2 (n^2 + 2)^2}{(4\pi)^3 (n^2 - 1)^{5/2}} \frac{\lambda^{7/2} \sqrt{2a}}{\sigma^2 B^2}. \quad (1.24)$$

The rms size of scatterers σ , and the correlation length of surface inhomogeneities B , are typically measured to be on the order of 2 nm and 5 nm respectively for glass surfaces. For excitation at $\lambda = 800$ nm in a 100 nm radius sphere this predicts a scattering limited Q-factor

$$Q_{s.s.} \approx 2 \times 10^9. \quad (1.25)$$

Measurements in fused silica optical fibers³⁴ demonstrate that the optical attenuation at $\lambda \approx 800 \text{ nm}$ is approximately 5 dB/km which corresponds to a material limit of

$$Q_{mat} = \frac{2\pi n}{\alpha \lambda} \approx 5 \times 10^9 \quad (1.26)$$

for fused silica microspheres. Such high Q WG modes have been demonstrated in large spheres ($a \sim 500 \mu\text{m}$) although it has also been shown that the Q degrades to 10^9 within five minutes of fabrication due to adsorption of water onto the microsphere surface³. On longer time scales, the layer of adsorbed water acts as a substrate for accumulation of microdust particles which further increases Q_{cont}^{-1} . Both of these degrading effects are partially reversible by reheating the sphere to bake off the contaminants. As a result, a single microsphere can be studied for months if necessary while maintaining $Q > 10^8$.

CHAPTER III

HAMILTONIAN DYNAMICS OF SYSTEMS WITH MIXED PHASE SPACE

Whispering gallery modes supported by fused silica microspheres are remarkable for their high Q factor and small mode volume. Despite these desirable properties their use in applications is limited considerably by the requirement for precisely positioned couplers due to a lack of directional emission. The isotropic emission of WG resonators is an obvious consequence of the rotational symmetry which enables the large Q factors. An interesting question is whether a dielectric resonator can be shaped so as to provide directional emission without destroying the attractive properties of the symmetric system. A variety of asymmetric dielectric cavity systems have been demonstrated that maintain some of the remarkable properties of their symmetric counterparts while providing highly directional emission²¹⁻²³. The term asymmetric resonant cavity (ARC) has been used to describe dielectric resonators whose properties are not derivable from those of the corresponding symmetric cavities. This is in distinction, for instance, from the slightly prolate spheroid where resonances can still be assigned mode numbers and frequency shifts can be calculated by perturbation theory.

The optical properties of ARCs can in principle be determined by solving Maxwell's equations for the relevant cavity shape. Analytical solutions only exist, however, for high symmetry cases such as a sphere or ellipsoid. Numerical wave solutions can be performed for two dimensional cavities but become computationally prohibitive for

cavities significantly larger than the optical wavelength (size parameter, $kR > 200$). The computational problem becomes even more dire when three dimensional cavities are considered. For ARCs with large size parameter it is expected that the geometrical optics limit ($kR \rightarrow \infty$) should provide important physical insight into the optical behavior. The geometrical optics picture of a dielectric optical cavity is nearly identical to a billiard problem of classical mechanics³⁵. A point particle undergoes repeated specular reflection off of a closed, hard walled boundary. The only required modification in the optical case³⁶ is that the particle is able to penetrate the wall with a certain probability depending on the angle of incidence and determined by Fresnel's equations corrected for curvature.

In order to investigate the behavior of asymmetric optical cavities in the geometrical optics limit one must be familiar with the methods of nonlinear dynamics. In this chapter, specific properties of integrable systems will be reviewed with emphasis on aspects which are particularly useful in the extension to near-integrable systems. The Poincaré surface of section will be presented as powerful technique for visualizing the phase space dynamics for near-integrable systems. The foundational KAM theorem and Poincaré-Birkhoff theorem will be discussed as well as the concept of the Arnold web. We will then review how these have been applied to the understanding of asymmetric optical resonators. More in depth treatments of nonlinear Hamiltonian dynamics can be found in several useful texts^{37,38}.

Integrable Systems

A N dimensional system is integrable if its Hamiltonian is separable into N independent equations of motion. These N equations can then be directly integrated to achieve an analytic solution for the motion of the system. In order for a Hamiltonian to be completely separable there must exist N conservation laws. In this thesis we will consider only time independent (autonomous) Hamiltonians, $H(\mathbf{p}, \mathbf{q})$, for which

$$\frac{dH(\mathbf{p}, \mathbf{q})}{dt} = 0 \quad (1.27)$$

making the total energy a constant of the motion,

$$H(\mathbf{p}, \mathbf{q}) = E. \quad (1.28)$$

This immediately implies that any autonomous Hamiltonian with one degree of freedom is integrable. In one dimension equation (1.28) can be solved such that the momentum is completely determined by the position coordinate and initial energy,

$$p = p(q, E). \quad (1.29)$$

Using this relation, p can be eliminated from Hamilton's equation

$$\dot{q} = \frac{\partial H}{\partial p} = \dot{q}(q, E) \quad (1.30)$$

so that it can be integrated to determine $q(t)$ and therefore $p(t)$. If there are greater than one degree of freedom then additional conservation laws must be found for the system to be integrable.

Conservation laws are direct consequences of symmetry in the system. It is possible that there can exist hidden symmetry in a system which produces conservation laws and make the system integrable. There is no general method, however, for determining the conserved quantities or even whether they exist. It is useful though to describe a set of conditions which will guarantee that a system is integrable. A canonical transformation $S(\mathbf{q}, \bar{\mathbf{p}})$ allows one to change from $\mathbf{q}, \mathbf{p}$ to a new set of canonical coordinates and conjugate momenta $\bar{\mathbf{q}}, \bar{\mathbf{p}}$:

$$\mathbf{q}, \mathbf{p} = \frac{\partial S(\mathbf{q}, \bar{\mathbf{p}})}{\partial \mathbf{q}} \leftrightarrow \bar{\mathbf{p}}, \bar{\mathbf{q}} = \frac{\partial S(\mathbf{q}, \bar{\mathbf{p}})}{\partial \bar{\mathbf{p}}}. \quad (1.31)$$

If a transformation can be found such that

$$H(\mathbf{p}, \mathbf{q}) \leftrightarrow H'(\bar{\mathbf{p}}) \quad (1.32)$$

then the system is integrable. In these new coordinates Hamilton's equations are

$$\begin{aligned} \dot{\bar{\mathbf{p}}} &= -\frac{\partial H'(\bar{\mathbf{p}})}{\partial \bar{\mathbf{q}}} = 0 \\ \dot{\bar{\mathbf{q}}} &= \frac{\partial H'(\bar{\mathbf{p}})}{\partial \bar{\mathbf{p}}} = \dot{\bar{\mathbf{q}}}(\bar{\mathbf{p}}). \end{aligned} \quad (1.33)$$

The first equation shows that the new momenta $\bar{\mathbf{p}}$ remain constant and represent the N conserved quantities that make the system integrable. This fact together with the second equation allows us to solve the motion

$$\bar{q}_i(t) = \mathbf{v}_i \cdot t + \bar{q}_{i0} \quad (1.34)$$

where $\mathbf{v}$ is determined entirely by $\bar{\mathbf{p}}$. One could make similar transformations using any N momenta which are linearly independent functions of $\bar{\mathbf{p}}$. If $\mathbf{p}$ is a periodic function of

$\mathbf{q}$ then it is particularly convenient to choose

$$\bar{p}_i = J_i \equiv \frac{1}{2\pi} \oint p_i dq_i \quad (1.35)$$

where the integral is over one period of the motion of $\mathbf{q}$. For this special choice of action-angle variables $\boldsymbol{\theta}, \mathbf{J}$ we have

$$\theta_i(t) = \omega_i \cdot t + \theta_{i0} \pmod{2\pi} \quad (1.36)$$

and ω_i is the radian frequency of the oscillation in θ_i .

The action-angle formulation is useful for visualizing the phase space motion of an integrable system. Consider a system with two degrees of freedom for which phase space has four dimensions. For an autonomous Hamiltonian, whether integrable or not, motion in phase space is confined to a 3D surface defined by conservation of energy. If the system is integrable the motion is further confined to a two-dimensional surface. Because θ_1 and θ_2 are periodic the phase space motion of an integrable system can then be envisioned as motion on the surface of a 2D torus. This picture of phase space motion can be generalized to motion on a N-dimensional torus for an integrable system with N degrees of freedom.

Figure 5 depicts motion on a torus for a system with two degrees of freedom. The angle θ_1 corresponds to the angular position around the cross section of the tube while θ_2 is the position along the closed tube. These tori are often referred to as invariant tori since for a particular energy the torus is parameterized by the conserved quantity J_1 . Choosing J_1 determines J_2 , ω_1 , and ω_2 . The winding number

$$W = \frac{\omega_1}{\omega_2} \quad (1.37)$$

determines the character of the motion on the torus. If $W = r/s \leq 1$, where r and s are integers with no common factors, then the path traced on the torus will close after r windings in θ_1 . There will be an infinite family of trajectories with $W = r/s$ and different initial θ . Together these trajectories will completely cover the 2D surface of the torus although each individual trajectory is confined to a 1D line drawn on the torus. The winding number can also be an irrational number $W \neq r/s$ and a single such trajectory will never close and cover the entire torus eventually approaching arbitrarily close to every point. A trajectory with irrational winding number is called quasiperiodic

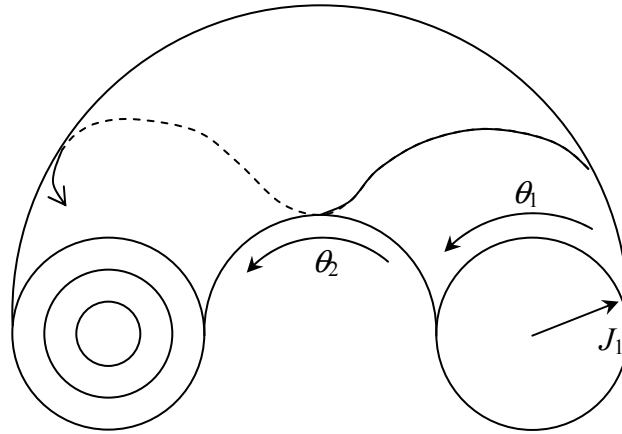


FIGURE 5. Phase space motion of an integrable system with two degrees of freedom occurs on a torus defined by the system energy and conserved action J_1 . The left cross section indicates concentric tori which have the same energy but different J_1 .

as it can be closely approximated by a periodic trajectory with large r and s . Motion of the system with different values of J_1 corresponds to nested tori with periodic and quasiperiodic tori densely intermingled. Quasiperiodic trajectories will play a crucial role in our understanding of the dynamics of systems which are perturbed from an integrable system.

Near-Integrable Systems and KAM Theorem

A near-integrable system is one which is related to an integrable system $H_0(\mathbf{J})$ by a small perturbation to the Hamiltonian which depends on the action-angle variables of the unperturbed system.

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + \epsilon H_1(\mathbf{J}, \boldsymbol{\theta}) \quad (1.38)$$

Standard perturbation theory leads to a series of terms with denominators of the form

$$\mathbf{n} \cdot \boldsymbol{\omega} = \sum_i n_i \omega_i \quad (n_i \text{ integers}). \quad (1.39)$$

Clearly, for any $\mathbf{J}$, an $\mathbf{n}$ can be found for which $\mathbf{n} \cdot \boldsymbol{\omega} = 0$ causing the perturbation series to diverge. In two dimensions it is apparent that this occurs when $W = r/s$. Thus all periodic orbits lead to small denominators and the breakdown of perturbation theory. It must be emphasized that this is not simply a mathematical failing but is the effect that actual resonances in the physical system have on the phase space topology.

Despite this inherent failing of perturbation theory, very useful results can be obtained for regions of phase space far enough removed from strong resonances. The

remarkable KAM theorem³⁹ guarantees that invariant tori will exist for a broad range of conditions. There are a number of technical conditions which must be met such as a sufficient number of continuous derivatives of H_1 . The primary condition of interest is that the winding number must be sufficiently irrational. This condition can be written for the two degree of freedom case as

$$\left| \frac{\omega_1}{\omega_2} - \frac{r}{s} \right| > \frac{k(\epsilon)}{s^{2.5}} \quad (1.40)$$

where $k(\epsilon \rightarrow 0) \rightarrow 0$. When the KAM conditions hold, the phase space motion will be confined to a torus which is somewhat perturbed from the invariant torus associated with the same irrational winding number in the integrable system. In the perturbed system $\mathbf{J}$ is no longer conserved although for the KAM tori a local invariant exists such that $\mathbf{J} = \mathbf{J}(\boldsymbol{\theta})$ describes the shape of the torus.

Equation (1.40) reveals two important features about near-integrable systems. First, the dependence of k on ϵ guarantees that a perturbation to an integrable Hamiltonian does not immediately cause all trajectories to become chaotic. In fact, KAM tori will fill large regions of phase space at small perturbations and gradually disappear as the perturbation grows. Second, the number s in the denominator demonstrates that low order resonances dominate the breaking of KAM tori. The existence of KAM tori in near-integrable systems with two degrees of freedom will be shown to have profound consequences on their overall dynamics.

The concept of phase space motion on a torus can be generalized to more degrees of freedom. Although it becomes difficult to visualize physically, the essential property is

that changing any of the angle coordinates by an integer multiple of 2π returns the system to its starting configuration. The existence of a KAM torus in a system with N degrees of freedom depends on having a trajectory for which there are N conserved quantities. This means that in a system of N degrees of freedom KAM trajectories will lay on N dimensional tori within the $2N-1$ dimensional “energy space” defined by conservation of energy. We have seen that the case $N = 1$ is special in that any autonomous Hamiltonian must be integrable. The case $N = 2$ is also special in that the two dimensional KAM tori define isolated closed volumes within the three dimensional energy space. In other words trajectories which start in the phase space region inside a KAM torus can never explore the phase space region outside of the torus. For $N \geq 2$ this is not the case. For instance in a system with three degrees of freedom, the KAM tori are three dimensional surfaces residing in the five dimensional energy space. These surfaces cannot isolate regions of the phase space any more than lines could isolate regions of a three dimensional space.

Poincaré Surface of Section

Although invariant tori are very useful in visualizing phase space motion and understanding the dynamics of near integrable systems they are not well suited to providing a complete map of phase space. The fact that invariant tori are objects in 3D space and nested inside one another makes it challenging to represent one and impossible to represent more than one in a two dimensional picture. Furthermore, invariant tori simply do not exist near resonances. The concept of a surface of section (SOS) developed by Poincaré⁴⁰,

provides a powerful map of phase space in two dimensions and is able to simultaneously depict multiple trajectories including those near resonances. The basic concept of a SOS is to reduce the dimensionality of phase space to a manageable two dimensions. This is accomplished by choosing a convenient surface in phase and recording the coordinates of each intersection of a trajectory with this surface. It is not immediately obvious that such a technique will provide a useful representation of phase space, and in fact the technique does depend on a wise choice of coordinates and surface.

Let us first consider a general autonomous Hamiltonian with two degrees of freedom expressed in action-angle coordinates

$$H(J_1, J_2, \theta_1, \theta_2) = E. \quad (1.41)$$

Conservation of energy confines trajectories to a 3D surface in the 4D phase space. We can solve equation (1.41) for J_2 in terms of the other three variables and the conserved energy.

$$J_2 = J_2(J_1, \theta_1, \theta_2, E) \quad (1.42)$$

The fact that J_2 can be calculated from the other variables allows us to project the phase space motion into a 3D space with coordinates J_1, θ_1, θ_2 without losing any information about the trajectories. In order to reduce the picture to two dimensions we choose a surface, e.g. $\theta_2 = \text{const.}$ and record the coordinates J_1, θ_1 each time the trajectory crosses the surface from a certain side. In this way a single trajectory will be represented as a sequence of points on the two dimensional SOS. Every point on the SOS corresponds to a unique point in the 4D phase space and therefore a unique trajectory.

This means that distinct trajectories can never cross and multiple trajectories can be plotted on a single SOS in order to provide a complete map of the system dynamics.

The SOS can also be thought of as a discrete time mapping (or twist mapping) of the phase space where the coordinates of each point is determined by the coordinates of the previous point. In an integrable system the mapping T in the $J_1 - \theta_1$ SOS is simply

$$T = \begin{cases} J_1^{(n+1)} = J_1^{(n)} \\ \theta_1^{(n+1)} = \theta_1^{(n)} + 2\pi W \end{cases} \quad (1.43)$$

where $J_1^{(n)}$, and $\theta_1^{(n)}$ are the coordinates of the n^{th} intersection with the surface of section.

For an integrable system the action variables are constants of the motion and each trajectory will consist of a set of points at a single value of J_1 . A trajectory with $W = r/s$ will have exactly s points and obey

$$T^s \begin{pmatrix} J_1^{(0)} \\ \theta_1^{(0)} \end{pmatrix} = \begin{pmatrix} J_1^{(s)} \\ \theta_1^{(s)} \end{pmatrix} = \begin{pmatrix} J_1^{(0)} \\ \theta_1^{(0)} \end{pmatrix}. \quad (1.44)$$

There will be an infinite family of such trajectories corresponding to all values of $\theta_1^{(0)}$. A trajectory with $W \neq r/s$ will have infinitely many points which would eventually fill a line $J_1(\theta_1) = \text{const.}$ For a near-integrable system a modified twist mapping T_ϵ exists and J_1 is not conserved. As we have seen, however, the KAM theorem guarantees that for small enough perturbations there will exist local invariants associated with trajectories having irrational winding number. We can express such a local conservation as

$$I(J_1, J_2, \theta_1, \theta_2) = \text{const.} \quad (1.45)$$

As before this allows us to solve for one variable in terms of the others, e.g.

$$J_1 = J_1(J_2, \theta_1, \theta_2) \quad (1.46)$$

Eliminating J_2 using equation (1.42) and evaluating at the chosen surface $\theta_2 = \text{const.}$ we can see that a KAM trajectory will produce a set of points filling the line

$$J_1 = J_1(\theta_1) \quad (1.47)$$

in the SOS. It should also be apparent that equation (1.47) would be unchanged if the chosen surface of section is defined by $\theta_2 = \theta_2(\theta_1)$ rather than $\theta_2 = \text{const.}$

In this dissertation we will also find it necessary to study a system with three degrees of freedom. The procedure for creating a SOS is very similar for more than two degrees of freedom although the result is not always quite as neat. With three degrees of freedom, conservation of energy confines trajectories to a 5D surface in the 6D phase space. As before this allows us to express one action, (say J_3) in terms of the other variables and project the motion into a 5D space. If we choose as our surface of section $\theta_3 = \text{const.}$ then we will have reduced our picture of the motion to a 4D space with coordinates $J_1, J_2, \theta_1, \theta_2$. Although this is an improvement over the original 6D phase space it is certainly not practical for plotting. One can imagine performing further sections to reduce the dimensionality. The problem is that, in general, this will not retain very many data points and most dynamical information would be lost. Instead, it is usually preferable to project the 4D motion onto each of the six unique 2D planes defined by pairs of the four remaining variable. This approach also has disadvantages. Although different trajectories can never have a point in the 4D reduced phase space in common, when projections are

made into 2D planes trajectories may appear to overlap. In addition, KAM tori can appear to occupy an area in the 2D plane rather than lying on a 1D curve despite the fact that they have no thickness in the 4D phase space. In practice one hopes to find that the dynamics of interest can be conveniently represented with less than six projections and minimum overlap confusion.

In a system with three degrees of freedom it is possible for symmetry to exist such that one action variable remains conserved without making the system integrable. The near-integrable Hamiltonian for this case would have the form

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + \epsilon H_1(J_1, J_2, J_3, \theta_1, \theta_2) \quad (1.48)$$

so that

$$\dot{J}_3 = \frac{\partial H}{\partial \theta_3} = 0. \quad (1.49)$$

This allows us to solve the motion in angle θ_3

$$\dot{\theta}_3 = \frac{\partial H}{\partial J_3} = \omega_3 = \text{const.} \quad (1.50)$$

Conservation of energy allows us to express J_2 in terms of the other variables (not including θ_3 which does not appear in the Hamiltonian). In this expression we can replace J_3 with its constant value leaving

$$J_2 = J_2(J_1, \theta_1, \theta_2). \quad (1.51)$$

This is formally identical to the case of two degrees of freedom and can therefore be reduced to a single 2D SOS plot. More importantly, this result allows us to state that for a system of three degrees of freedom, if the motion in one coordinate is separable then

the dynamics can be effectively reduced to that of a corresponding system with only two degrees of freedom. The qualitative differences between dynamics in systems of two and three degrees of freedom give added significance to this result.

Two Dimensional Integrable Billiards

Let us start by considering a circular, hard-wall billiard with radius R . The Hamiltonian for the circular billiard expressed in polar coordinates is

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right). \quad (1.52)$$

where p_r and p_ϕ are the radial and tangential momenta respectively. Due to the symmetry ϕ does not appear in the Hamiltonian and therefore p_ϕ is conserved. We can convert to action-angle variables by defining

$$J_\phi = \frac{1}{2\pi} \int_0^{2\pi} p_\phi d\phi = p_\phi \quad (1.53)$$

and

$$J_r = \frac{2}{2\pi} \int_{r_0}^R p_r dr \quad (1.54)$$

where $r_0 = \sqrt{2mE}$ is the closest the particle approaches the center and the extra factor of two accounts for the radial motion out to the wall and back. The action J_r can be determined in terms of r_0 , R and p_ϕ but we need not do this explicitly in order to

construct a Poincaré surface of section. It is enough to know that

$$J_r = J_r(J_\phi, \theta_r, \theta_\phi) \quad (1.55)$$

so that we may project the phase space motion into a 3D space. We also need not worry about the exact form of θ_r as long as we make the convenient choice of surface of section corresponding to the walls of the billiard. For the circular billiard this surface is obviously defined by $\theta_r = \text{const.}$ where the constant is determined by converting $r = R$ into the new angle variable. For a general shape the surface $r = r(\phi)$ could be converted into $\theta_r = \theta_r(\theta_\phi)$. In both of these cases we are then able to plot the surface of section with coordinates θ_ϕ and J_ϕ . In the circle we have seen that this corresponds simply to ϕ and p_ϕ .

More useful physical insight regarding optical cavities can be gained by plotting ϕ and $\sin \chi$ where χ is the angle of incidence measured from the normal to the boundary. In the circular billiard this amounts to simply scaling p_ϕ by the linear momentum. The value $\sin \chi$ can also be related to the winding number as $\sin \chi = \cos(\pi W)$ in a circular billiard. Figure 6 shows a SOS for a circular billiard with a number of different trajectories. The symbols represent periodic trajectories with rational winding number.

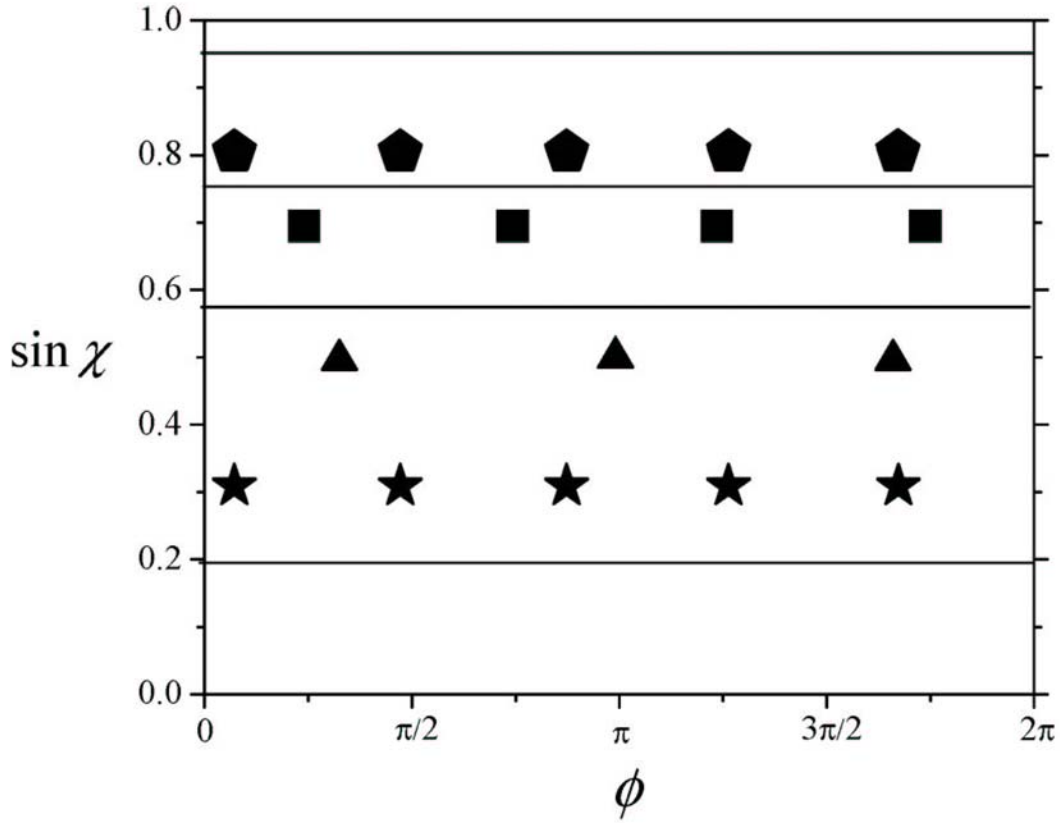


FIGURE 6. Poincaré surface of section for the circle billiard. Shapes indicate periodic orbits with a corresponding real space pattern. Horizontal lines indicate quasiperiodic trajectories that eventually visit every point on the billiard wall.

The symbol shape indicates the real space pattern of the closed orbit. Each periodic orbit is just one of an infinite family of congruent orbits which can be formed by any amount of rotation about the center of the circle. Notice that the pentagon ($W = 1/5$) and five point star ($W = 2/5$) orbits have reflections at the same locations on the billiard but are distinguishable in the SOS by different values of $\sin \chi$. In addition to the periodic orbits several quasiperiodic orbits are indicated by solid lines at $\sin \chi = \cos(\pi W) = \text{const.}$ where $W \neq r/s$.

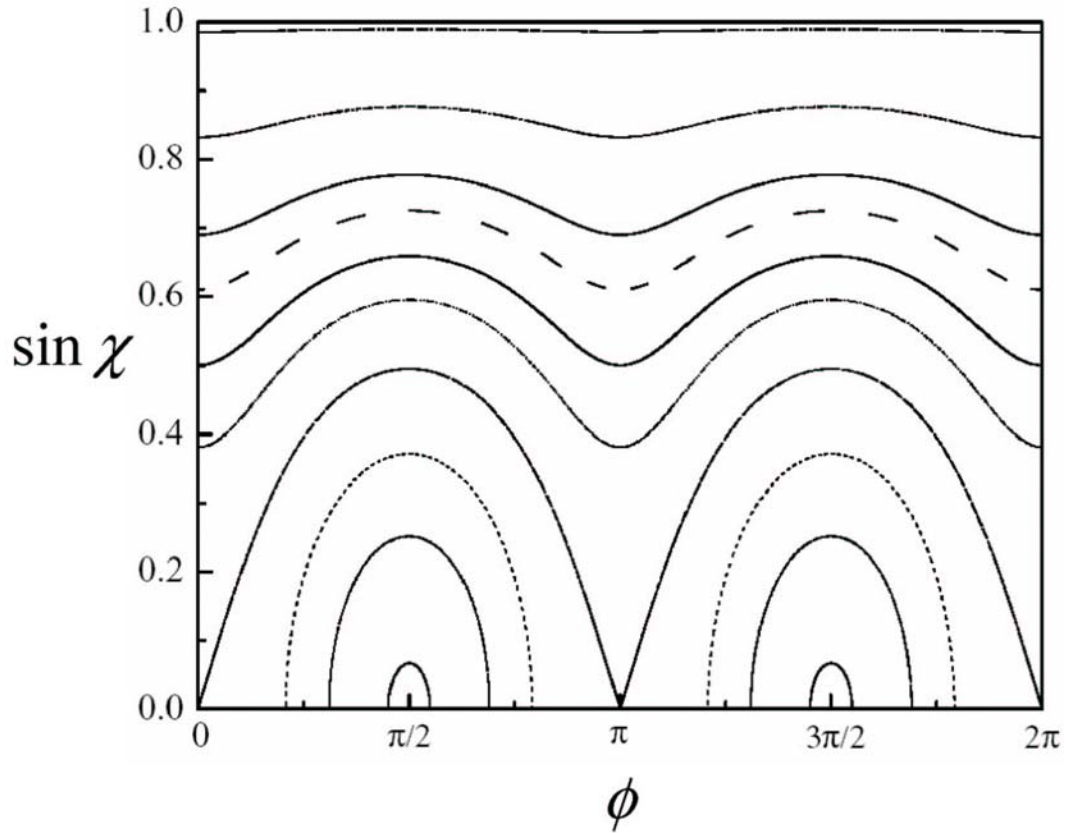


FIGURE 7. Poincaré surface of section for an ellipse billiard with $\epsilon = 0.07$. All trajectories are periodic or quasiperiodic. Periodic trajectories are neutrally stable and do not form islands. The one exception is the two bounce orbit which is stabilized along the short axis and destabilized along the long axis. The separatrix trajectory, which touches $\sin \chi = 0$ at $\phi = 0, \pi, 2\pi$, results from the destabilized two bounce orbit.

In non-circular billiards $\sin \chi$ is proportional to the tangential momentum p_t rather than p_ϕ . A proper SOS in this case would then be formed using the so-called Birkhoff coordinates, s and p_t where s , the arc length along the boundary, is the conjugate variable to p_t ³⁵. As long as the cavity is everywhere convex, there will be a

one to one correspondence between s and ϕ allowing us to still use ϕ and $\sin \chi$ for all SOS plots. The fact that ϕ and $\sin \chi$ are not actually conjugate variables for asymmetric cavities means that the SOS will not have the property of area preservation. This should be kept in mind but will not be an issue in this dissertation.

The ellipse is a unique perturbation of the circle which maintains integrability. It turns out that for any trajectory in an elliptical billiard the product of the angular momenta about each foci is a conserved quantity. This constant of the motion causes every trajectory to lay on an invariant tori. When a SOS is plotted for the ellipse (fig. 7) using the action-angle variables of the circle the trajectories no longer form horizontal lines. Instead each quasiperiodic trajectory eventually fills a curve $p_i = p_i(\phi)$. This amounts to the angle of incidence being modified at each location in accordance with the new curvature at that point. Infinite families of periodic trajectories remain as in the circle. The exception to this is the two bounce periodic trajectories of which only two remain. The one along the short axis is stabilized by the elliptical deformation so that nearby orbits circulate around it. The one along the long axis is destabilized producing a separatrix which separates the island trajectories from the orbiting trajectories.

Two Dimensional Asymmetric Billiards

We now turn our attention to how the SOS will be modified by introducing a perturbation to the Hamiltonian. For hard walled billiards perturbation of the Hamiltonian

amounts to a change of shape of the boundary. Whereas the circular billiard is defined by $r(\phi) = R$, an asymmetric billiard can be expressed by

$$r(\phi) = R(1 + \epsilon \cdot d(\phi)) \quad (1.56)$$

where $d(\phi)$ is a function describing the deformation and is subject to the condition

$$d(0) = d(2\pi). \quad (1.57)$$

Certain deformations of the circle are more conveniently expressed in other forms (for example the ellipse or a stadium shape) but this dissertation will be primarily concerned with deformations easily expressed in the form (1.56). As in the circular billiard the value of R has no effect on the dynamics of the system and will typically be set to one. In this dissertation, our primary focus will be on the case of quadrupolar deformation described by

$$r(\phi) = 1 + \epsilon \cos(2\phi) \quad (1.58)$$

The quadrupole is lowest order deformation when $d(\phi)$ is expressed as a series of cosines with integer coefficients of ϕ . The dipole deformation ($d(\phi) = \cos(\phi)$) is predominantly a shift of center of mass with the dominant shape change being quadrupolar. The ellipse can be expressed in the form of (1.56) where $d(\phi)$ is an infinite series of even order multipoles. In the ellipse the dominant term is quadrupolar although the dynamics of the exact shape is qualitatively different from that of any other deformation from the circle in that the ellipse remains integrable even with arbitrarily large eccentricity.

Lazutkin Theorem

The KAM theorem guarantees that tori associated with sufficiently irrational winding number will not be destroyed by a small perturbation to the Hamiltonian. The irrationality condition (1.40) indicates that the tori with the most irrational winding number will be the last to be destroyed as the perturbation strength is increased. The most irrational number, called the *golden mean*, is the solution to the quadratic equation $x^2 - x - 1 = 0$. This solution

$\gamma = \frac{1}{2}(\sqrt{5} + 1) \approx 1.618$ is the number least well approximated by a fraction with a denominator

smaller than any chosen value. Only winding numbers less than one occur in billiard dynamics meaning that $W = 1/\gamma \approx .618$ would be expected to be the last KAM curve destroyed by

increasing ϵ . This winding number corresponds to $\sin \chi = 0.36$ in a circle so we would

expect to find a KAM curve in this region of the SOS for all small deformations of the circle.

Figure 8, showing the SOS for a quadrupole with $\epsilon = 0.065$, demonstrates that this is clearly not the case since KAM curves are visible at large $\sin \chi$ but not near $\sin \chi = 0.36$. This

effect in billiards was studied by Lazutkin who proved that families of KAM curves would

persist near $\sin \chi = 1$ as long as the billiard remains everywhere convex⁴¹. This effect has been

explained as arising due to the fact that the effective perturbation strength in a billiard is a local quantity depending on the value of $\sin \chi$ ⁴². One can, therefore, understand the condition for

the existence of KAM tori in a billiard by replacing $k(\epsilon)$ in (1.40) with $k(\epsilon, \sin \chi) \rightarrow 0$ for

$\epsilon \rightarrow 0$ or $\sin \chi \rightarrow 1$. This important result guarantees that whispering gallery type modes will exist in any completely convex billiard.

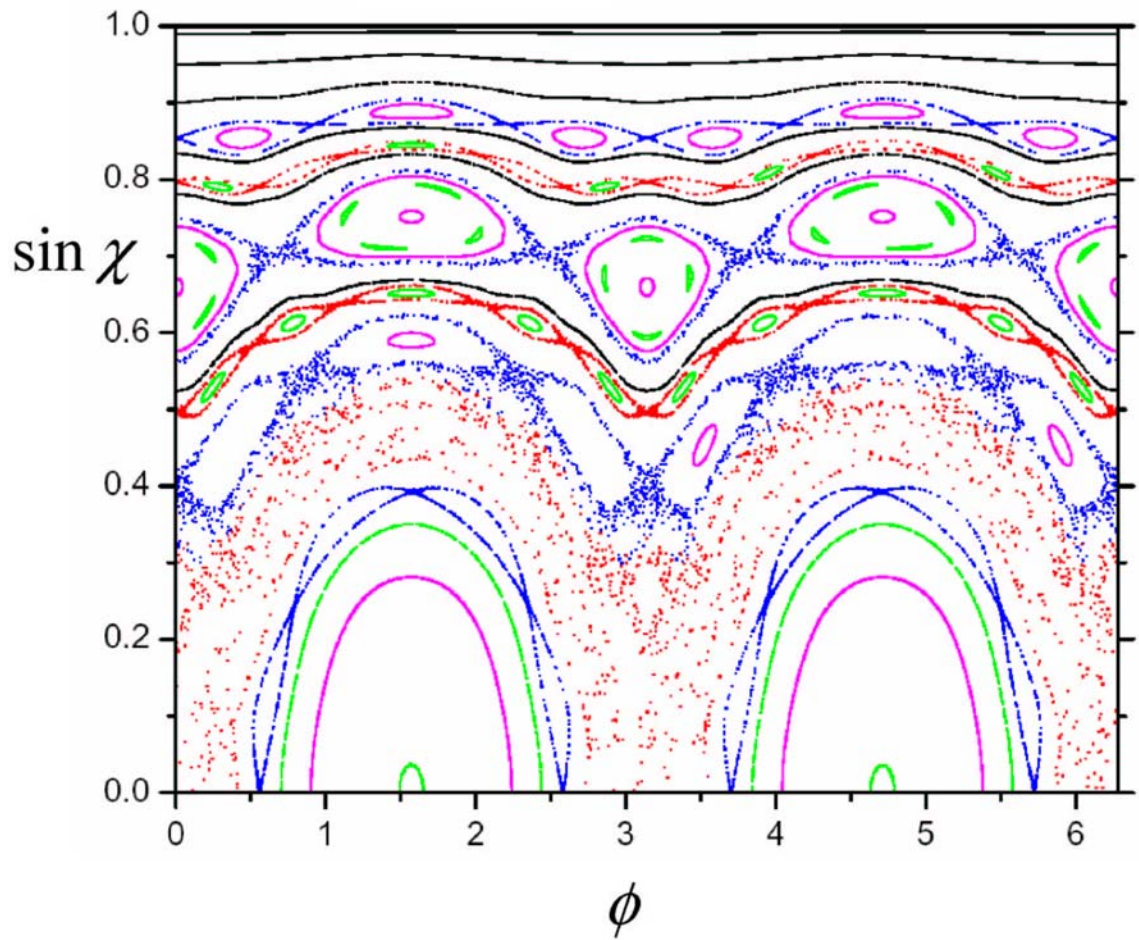


FIGURE 8. Poincaré surface of section for a quadrupole billiard with $\epsilon = 0.065$. KAM curves are plotted in black. Stable islands are plotted alternately in green and magenta. Chaotic trajectories are plotted alternately in red and blue.

Poincaré-Birkhoff Theorem

To this point we have considered only the effect that a perturbation has on tori with irrational winding number. We must also consider the effect of a perturbation on trajectories which are periodic in the integrable system. In the circular billiard periodic trajectories are neutrally stable. This means that the distance in phase space between the relevant trajectory and one with a slightly different initial condition will increase linearly. Clearly, if the starting position of a periodic orbit in the circle billiard is changed but the initial angle of incidence is unchanged the distance between the two trajectories will remain constant. If on the other hand, the angle of incidence is changed by an amount $\Delta\chi$ then the distance between the two trajectories $\Delta\phi$ will grow by $2\Delta\chi$ with each reflection. When the circular billiard is deformed four things can happen to any particular periodic orbit. A periodic trajectory can be destroyed, destabilized, stabilized, or simply rearranged as in the ellipse.

The Poincaré-Birkhoff theorem⁴³ provides some constraints on which of these possibilities occur. The theorem states that when a rational torus ($W = r/s$) is perturbed there will remain $2ks$ fixed points for some integer k . Furthermore, there are two types of fixed points, elliptic (stabilized) and hyperbolic (destabilized), which are arranged alternately in the angle coordinate. The integer k is typically small, can be different for different W and is determined by symmetry. We can understand the origin of elliptic and hyperbolic fixed points in billiards by considering a SOS.

Let us consider the effect that a perturbation has on the trajectories with $W = 1/4$ circulating counterclockwise (each reflection occurs at a larger value of ϕ) in the circular billiard. These trajectories are at $\sin \chi = \sqrt{2}/2$ and repeatedly visit the same four points in phase space. Each point on the line $\sin \chi = \sqrt{2}/2$ will therefore remain stationary under the mapping T^4 . Figure 9a shows how near the periodic orbits there are trajectories with $\sin \chi > \sqrt{2}/2$ for which the mapping T^4 causes ϕ to decrease and trajectories with $\sin \chi < \sqrt{2}/2$ which increase ϕ under T^4 . When the perturbation is introduced, the invariant tori will be deformed or broken but the remnants of the rational torus will still have KAM curves above and below which decrease and increase ϕ respectively under the mapping T_ϵ^4 . Between these two KAM curves there must be a solid curve, as shown in figure 9b, which has the property $T_\epsilon^4(\phi) = \phi$. Each point on this curve will then map to a point on a new curve (red curve in fig 9c) with the same ϕ but possibly different $\sin \chi$. In a proper SOS, using conjugate coordinates, these two curves must have the same area under them. This condition requires that the two curves cross each other an even number of times. This feature persists in our $\phi - \sin \chi$ SOS as well. The crossing points clearly satisfy

$$T_\epsilon^4 \begin{pmatrix} \sin \chi_0 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} \sin \chi_0 \\ \phi_0 \end{pmatrix} \quad (1.59)$$

and are thus the $2ks$ fixed points ($k = 1$ in a quadrupole as shown). This is called a primary resonance since it directly corresponds to a periodic orbit which existed in the

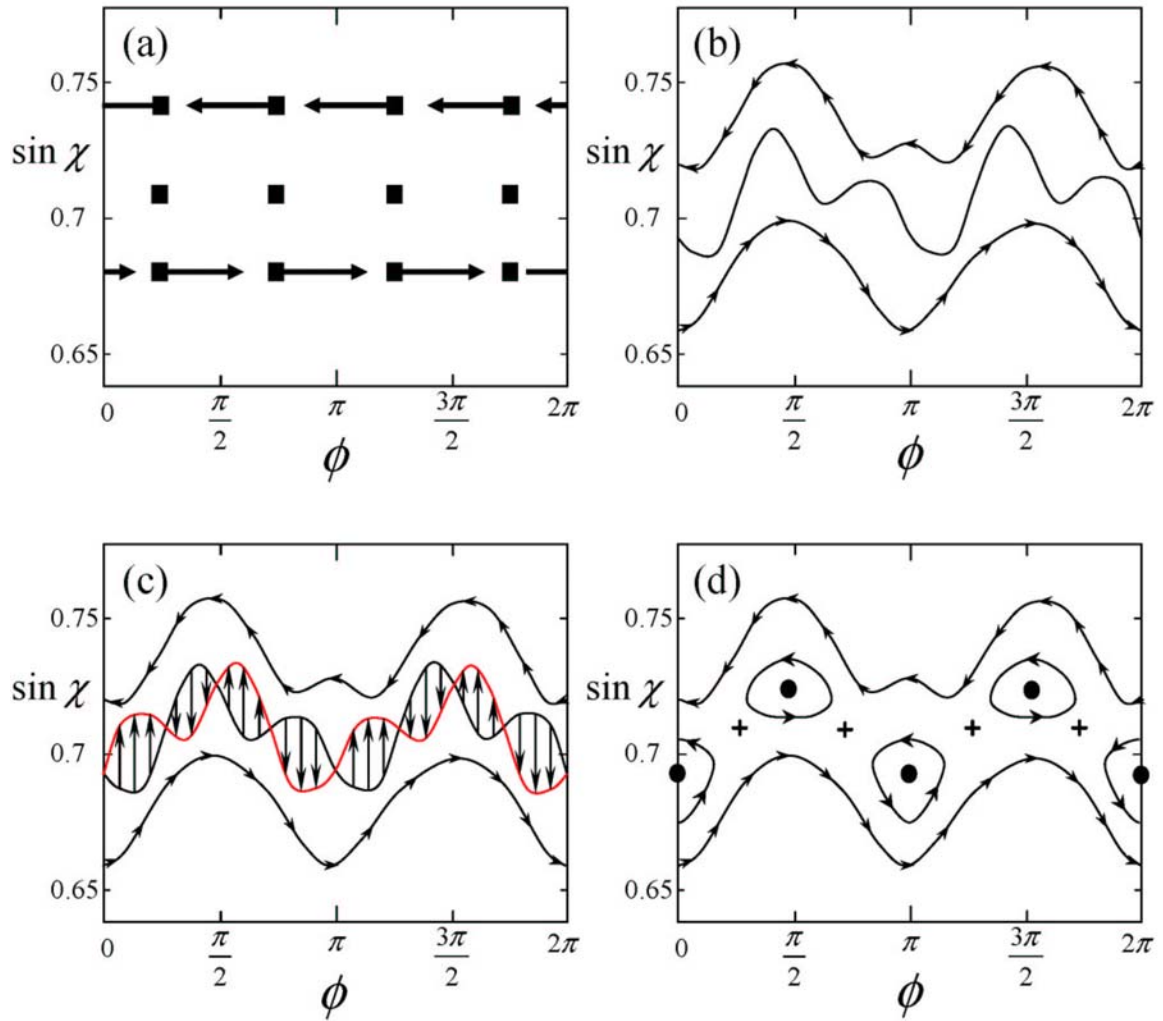


FIGURE 9. Poincaré-Birkhoff theorem. (a) A four bounce periodic trajectory of the circle billiard is shown with nearby quasiperiodic orbits. (b) Two KAM curves of a perturbed billiard are shown. Between these curves is the set of points which do not change ϕ under the T^4 mapping. (c) The red curve represents the set of points produced by one T^4 iteration applied to the curve in (b). (d) Elliptic (circles) and hyperbolic (crosses) fixed points are indicated. Stable islands form around the elliptic fixed points. (a-d) The arrows indicate how each point evolves under the T^4 mapping.

integrable system. Examining the flow of points as they are mapped by T_ϵ^4 we see that two types of fixed points exist alternately along the ϕ direction. The flow near half of the fixed points closes into a counterclockwise loop. These elliptic fixed points are plotted as circles in figure 9d along with a trajectory which forms a chain of stable ‘islands’ circling the fixed points in a counterclockwise rotation. Near the other fixed points, called hyperbolic and plotted as crosses, the flow tends to bring trajectories towards the fixed point along two directions and then away following the flow on the nearby KAM curves. There are no trajectories that remain close to the hyperbolic fixed points for extended periods of time as there are around elliptic fixed points.

The dynamics in the vicinity of an elliptic fixed point is quite complex. A low order primary resonance ($W = 1/s$) strongly effects the topology of the tori which do not satisfy (1.40). Tori in this region which are sufficiently irrational to satisfy a modified version of (1.40) form solid curves encircling the elliptic fixed points. These curves correspond to KAM tori which intersect the SOS s times and appear as stable islands around the periodic points in the SOS (figure 9d). In addition, the perturbation produces secondary resonances around the s bounce fixed point. These secondary resonances have $W = nr/ns$ and form chains of elliptic and hyperbolic fixed points which encircle the four original elliptic points. These secondary resonances are also surrounded by islands and tertiary resonances. Thus there exists a self similar pattern of KAM lines and fixed points at ever finer scales in the SOS.

The dynamics near the hyperbolic fixed points is in some ways even more complicated. The k_s hyperbolic points are surrounded by a separatrix layer which is bounded between the nearest KAM curves and excluded by the stable islands surrounding the elliptic points. The flow of a trajectory in this region can follow either of the bounding KAM curves or the circular flow around the islands and can gradually move between these different patterns of flow. For each hyperbolic point, four curves which guide the motion can be identified. Two are incoming trajectories and two are outgoing. These stable and unstable manifolds are invariant curves of the mapping T_ϵ^s . As a result, each curve is never allowed to intersect itself and yet is required to form an infinite number of intersections (called homoclinic points) with a curve of the opposite type. This infinitely complicated structure, often called the homoclinic tangle, leads to chaos, in the sense of a sensitive dependence on initial conditions. Any trajectories in a separatrix layer will gradually fill a finite area in the SOS. A single trajectory in a chaotic separatrix layer associated with a $W = 1/4$ resonance is shown in figure 10. Although the islands and KAM curves which define its extent are not shown, their effect is readily apparent.

Surface of Section for Quadrupole

Let us now examine the overall phase space of a quadrupole ($\epsilon = 0.065$) as depicted in figure 8. Individual trajectories in the figure are color coded to distinguish the three basic types of trajectories present in the deformed billiard. KAM trajectories produce

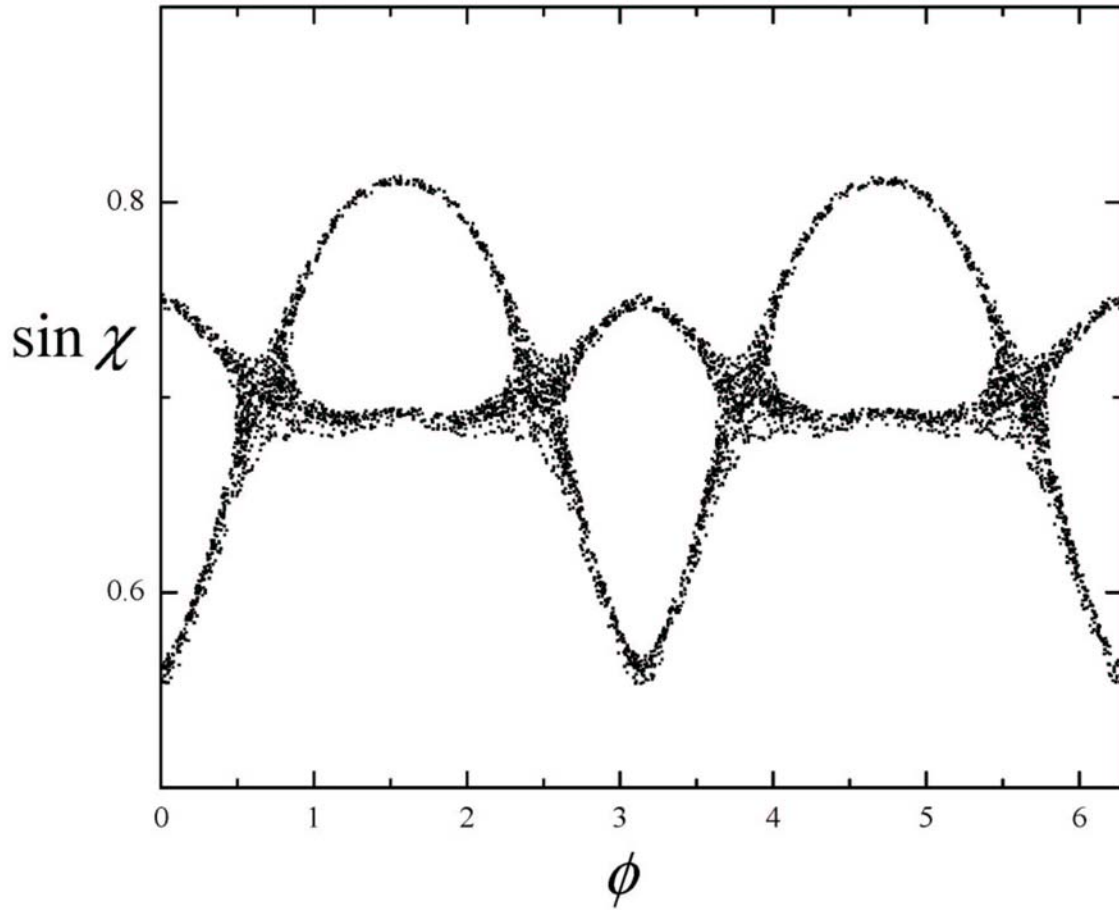


FIGURE 10. A single trajectory in the separatrix layer associated with a $W = 1/4$ resonance. The 2D area explored by the trajectory is bounded by KAM curves and stable islands.

an unbroken wavy line that completely traverses the SOS. The KAM curves shown for this quadrupole are plotted in black. Similar to the KAM curves are the stable islands which appear as a series of solid rings around elliptic fixed points. Individual island trajectories are plotted alternately as green and magenta in the figure in order to distinguish which ones correspond to the same chain. Chaotic trajectories in the

separatrix regions surrounding the islands are the only trajectories which fill a finite area in the SOS. Individual chaotic trajectories are plotted in red and blue alternately to distinguish trajectories with different initial conditions.

KAM curves dominate the top quarter of this SOS. As predicted by the Lazutkin theorem the KAM curves with $\sin \chi \approx 1$ are only slightly deformed from the horizontal lines in the SOS for the circle. At smaller values of $\sin \chi$ the KAM curves are strongly deformed due to the changing curvature around the billiard as in the ellipse and also due to the presence of stable periodic orbits which are not present in the ellipse. As discussed above the KAM curves act as impenetrable barriers confining the chaotic trajectories. No trajectory can have points in the SOS both above and below a single KAM curve.

The island structure in this SOS provides an interesting demonstration of the Poincaré-Birkhoff theorem. The theorem states that after the perturbation is applied there will remain ks elliptic and ks hyperbolic fixed points. Examining the primary resonances in the SOS for the quadrupole reveals that $k = 1$ for even s and $k = 2$ for odd s . Examples of even s resonances in the SOS are the six bounce orbit near $\sin \chi = 0.85$, the four bounce orbit near $\sin \chi = 0.7$, and the two bounce orbit near $\sin \chi = 0$. A higher order primary resonance is also shown near $\sin \chi = 0.6$. This ten bounce orbit forms a ten point star in real space and corresponds to $W = 3/10$. Each of these chains has exactly s islands which are all visited by a single trajectory. Also shown in fig. 8 are examples of secondary resonances around the two and four bounce orbits. These are periodic orbits which did not exist in the circular billiard and their appearance is an example of a special type of

bifurcation that occurs in billiards with high symmetry. The four small islands within each of the primary four bounce islands correspond to $W = 4/16$. For these secondary resonances k is typically 2 regardless of s . Example of odd s resonances are the five bounce orbit near $\sin \chi = 0.8$ and the three bounce orbit near $\sin \chi = 0.5$. In each of these cases the chaotic separatrix trajectory clearly surrounds $2s$ islands. Only s of these islands is visited by any single trajectory as shown for each case.

The property of the quadrupole which determines k for even and odd s primary resonances is the reflection symmetry across the x axis and y axis. Even s periodic orbits can share the symmetry of the quadrupole whereas odd s periodic orbits can only have a single reflection symmetry axis. This means that any odd s periodic orbit which is stabilized by the deformation will have one symmetry axis but there must also exist a sister periodic orbit which is obtained by reflecting the first across its non-symmetric axis. The symmetry of the billiard also affects the size of the islands. In general, islands decrease in size as s increases. This fact accounts for s appearing in the denominator on the right hand side of (1.40). We notice however in figure 8 that the three and five bounce islands are smaller than the four and six bounce islands respectively. Resonances which share the symmetry of the perturbed Hamiltonian are stronger than those which do not. These effects in the quadrupole are highlighted by comparison to the dipole (figure 11) deformation which has only one symmetry axis. In the SOS for a dipole $k = 1$ for all primary resonances. Also, the size of islands decreases with increasing s as expected since both even and odd s periodic orbits can match the symmetry of the billiard.

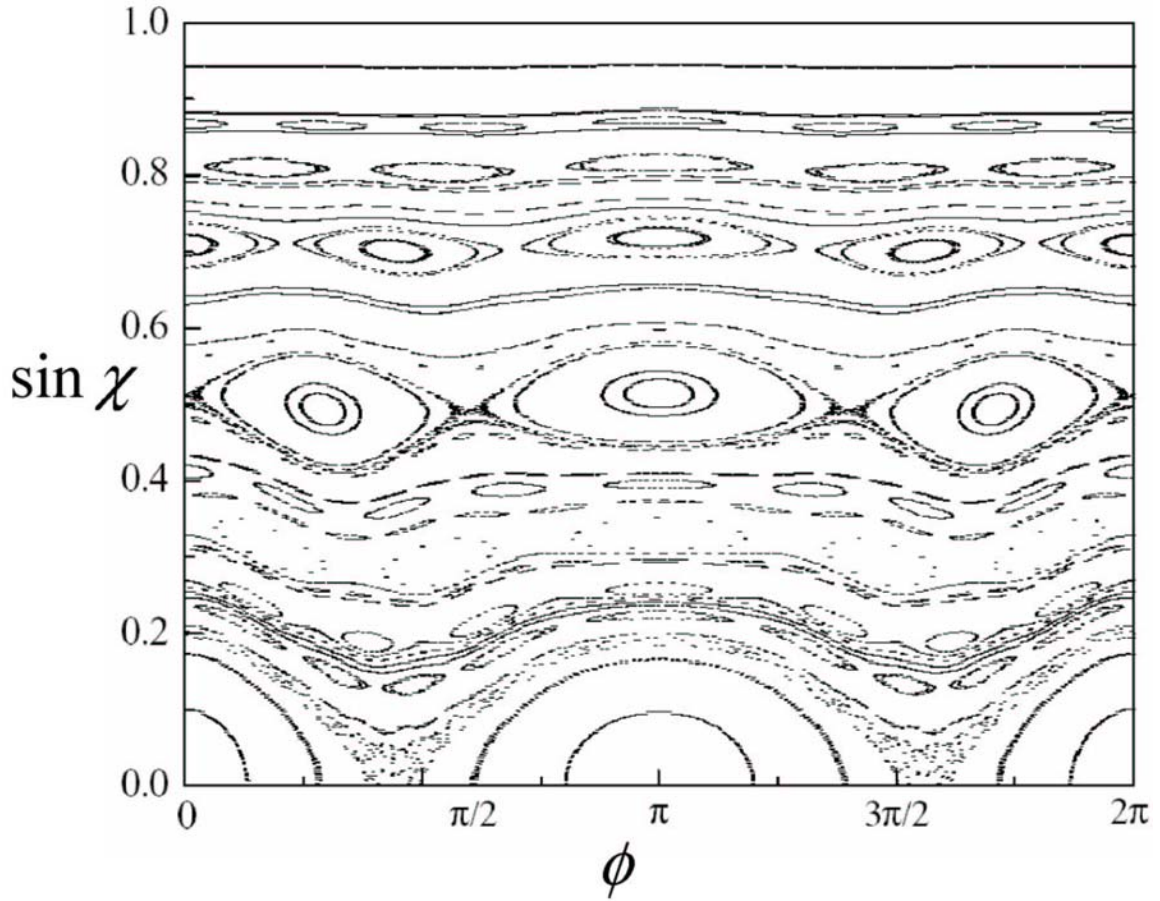


FIGURE 11. Poincaré surface of section for a dipole billiard with $\epsilon = 0.2$. The dipole only has one reflection symmetry axis and therefore shows no preference for even or odd periodic orbits. The size of the major stable islands decreases with increasing period.

There are large regions of the SOS in figure 8 where KAM curves do not exist. The separatrix layers which contain the hyperbolic fixed points grow and join as the deformation is increased and KAM tori are destroyed by overlapping resonances. The chaotic motion in each layer gradually fills the region bounded by KAM curves and islands eventually approaching arbitrarily close to every point in the region. The presence of stable

island trajectories which are not plotted is demonstrated by the “holes” in the chaotic region. Even when separatrix layers merge, trajectories can follow the topology of one separatrix for many reflections before diffusing towards motion along another. This can be seen in figure 8 by the slight overlap of the chaotic trajectories surrounding the two bounce and three bounce islands. If these trajectories are plotted for longer times they both fill the same region which surrounds both sets of islands.

Phase Space Transport

A chaotic trajectory in a two dimensional billiard will eventually visit a point arbitrarily close to every point in phase space limited only by the condition that it must not cross a KAM curve or stable island. KAM curves and islands represent the intersection of KAM tori with the surface of section. In a 2D SOS, any trajectory which has points in the SOS both above and below (inside and outside) a KAM curve (island) must have an intersection with the KAM torus at some point in phase space. Every point on a KAM torus must, by definition, remain on the KAM torus for all time. This means that no trajectory may intersect a KAM torus and therefore no trajectory may have points in a SOS on two sides of a KAM curve or island. In two dimensional billiards this condition strongly limits motion in phase space. This is especially true for small perturbations.

As the perturbation strength is increased KAM curves which no longer satisfy (1.40) for the larger ϵ are destroyed. The process of destruction of a KAM curve begins with the appearance of tiny holes at very fine scales in the form of a Cantor set⁴⁴. Such a

holey torus structure is given the appropriate name cantori. The holes grow until nothing remains of the KAM curve but chains of many tiny islands associated with the rational approximates to the irrational winding number. This complicated fine structure can “capture” a chaotic trajectory for long periods of time causing it to follow closely a nearby island structure or KAM curve. These cantori, remnants of KAM curves, can also slow the diffusion of trajectories from one separatrix layer to another when KAM curves no longer prevent such motion.

Increasing the perturbation strength causes chaotic layers to grow and merge to form large regions where chaotic diffusion is not limited by KAM curves. Although trajectories in these regions exhibit a strong sensitivity to initial conditions, the motion is definitely not random. The Hamiltonian system remains deterministic. Patterns of flow can be observed within the chaotic regions as pointed out above regarding figure 8. A good way to describe these patterns is by considering adiabatic invariant curves⁴⁵.

The usefulness of adiabatic invariant curves (ICs) arises from a difference in time scales for motion in the action and angle coordinates. For instance, a chaotic trajectory in a quadrupolar billiard will typically make many trips around the billiard with only a small change in $\sin \chi$ at any position ϕ . Although no exact invariant exists for trajectories in a chaotic region they will tend to follow an approximate IC which may slowly change. Because the quadrupole shape is so similar to the ellipse, the actual invariant curves of the ellipse provide a convenient analytic expression that describes the basic pattern of the chaotic trajectories on intermediate time scales between the fast ϕ motion and the slower

changes in $\sin \chi$. Approximation of the motion by ICs of the ellipse essentially amounts to ignoring the presence of periodic orbits.

As a trajectory approaches an island chain, insight into the dynamics can be obtained by considering the mapping T_ϵ^s as we did when discussing the Poincaré-Birkhoff theorem. For instance, as a trajectory approaches the four bounce island chain from above the T_ϵ^4 mapping reveals a flow across the SOS from large to small ϕ . This flow slows as the trajectory approaches the separatrix. Approaching the four bounce separatrix from below one observes the same effect but with the direction of flow reversed. In the region between the separatrix and an island the T_ϵ^4 mapping produces a counterclockwise flow of the trajectory. Motion along the separatrix has an infinite period in that it takes an infinite number of reflections to approach the hyperbolic fixed point along the separatrix. This is due to the infinitely complex tangle of the stable and unstable manifolds. As discussed above, these manifolds are invariant curves of the T_ϵ^4 mapping and thus act similarly to KAM curves. The difference is that the complex folding of these manifolds causes them to intersect with the manifolds associated with nearby higher order periodic orbits. The intersections, called heteroclinic points, allow trajectories to move across a separatrix making separatrices weaker barriers to phase space transport than KAM curves or even cantori. The heteroclinic points are concentrated near the hyperbolic fixed points meaning that transport across separatrices tends to occur near the hyperbolic points.

Arnold Diffusion

In systems with three degrees of freedom there are two new effects which dramatically affect phase space transport. First, we have already shown that in three degrees of freedom the KAM tori will be three dimensional surfaces residing in a five dimensional energy space. This means that, unlike the two degree of freedom case, KAM curves are not able to partition the available phases space into noncommunicating closed volumes. The second new feature is that resonances are no longer isolated from one another by conservation of energy. Together these two effects cause all chaotic regions of phase space to be connected. This allows a chaotic trajectory to come arbitrarily close to any point in phase space consistent with conservation of energy. This global transport phenomenon is called Arnold diffusion and the dense structure of chaos is the so-called Arnold web⁴⁶.

The difference in resonance behavior for systems of two and three degrees of freedom can be understood geometrically in action space. Let us consider a simple, N degree of freedom, integrable Hamiltonian in action-angle coordinates

$$H = \frac{1}{2} \mathbf{J}^2. \quad (1.60)$$

For $N = 2$ conservation of energy defines a circle in the $J_1 - J_2$ plane, whereas for $N = 3$ it defines a sphere in the $J_1 - J_2 - J_3$ space. As established earlier, a resonance occurs when

$$\mathbf{n} \cdot \boldsymbol{\omega}(\mathbf{J}) = 0 \quad (1.61)$$

where $\mathbf{n}$ is a vector with integer components and in this case $\omega_i = J_i$. In action space the resonance condition (1.61) defines lines with rational slope and zero intercept for $N = 2$. Figure 12a shows how each resonance is localized to a pair of points in the action plane and although resonances are dense on the constant energy circle they can not overlap. The $N = 3$ case is quite different. The resonance condition now defines planes which contain the origin as depicted in figure 12b. The intersections of these planes with the

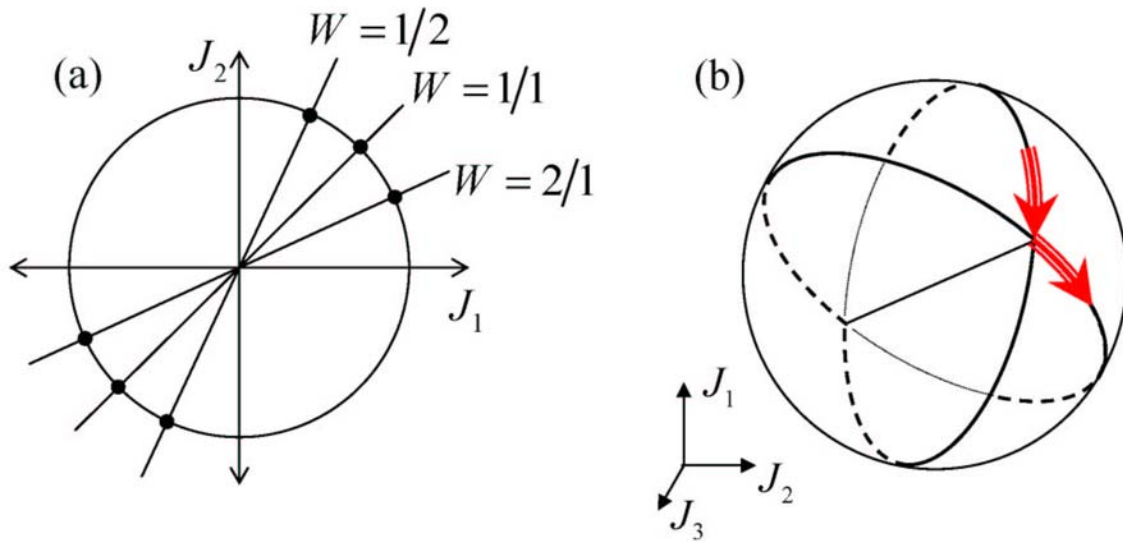


FIGURE 12. Resonances for two and three degrees of freedom. (a) In a system with two degrees of freedom, conservation of energy defines a closed curve in the $J_1 - J_2$ plane. This curve is intersected at discrete points by lines representing resonance conditions. (b) In a system with three degrees of freedom, conservation of energy defines a closed surface in the $J_1 - J_2 - J_3$ space. Resonance conditions define planes which intersect the constant energy surface on one dimensional curves. These resonance curves intersect allowing energy conserving motion from one resonance to another as indicated by the red arrows.

constant energy sphere are great circles. This demonstrates that in a system with three degrees of freedom any particular resonance can be achieved with a range of values for each action variable. Furthermore, the resonance circles must all intersect one another meaning that many points on the constant energy circle correspond to more than one resonance. In a near-integrable system where the actions are not constants of the motion a trajectory can then move along one resonance to an intersection and leave on a different resonance (red arrows in fig. 12b). This is the genesis of the Arnold web which connects all of the chaotic separatrix layers associated with resonances.

Although the above discussion reveals why Arnold diffusion occurs in systems with three degrees of freedom, it is an exceedingly difficult phenomenon to study quantitatively. Since it was first described by Arnold in 1964 it has primarily been studied numerically⁴⁷. Only for the simplest cases have these results been compared to theoretical models⁴⁸. Arnold's conjecture that global diffusion occurs in any near-integrable system with three or more degrees of freedom appears to be accurate although no rigorous proof exists. It is known that there is no critical perturbation strength required for Arnold diffusion although the diffusion rate goes to zero as the perturbation decreases. Arnold made an additional prediction that for systems with $N \geq 3$ degrees of freedom trajectories are always either on a KAM torus or in the Arnold web. This prediction, which has been bourn out by numerical studies⁴⁷, indicates that for any initial condition there are exactly $N-1$ or zero conserved quantities in addition to the energy. In other words, a trajectory must be periodic, quasiperiodic, or chaotic in all variables.

Arnold diffusion is extremely slow compared to the chaotic motion across a separatrix layer which occurs similarly to a two degree of freedom system. This leads to an additional adiabatic approximation when Arnold diffusion occurs. In order to visualize Arnold diffusion, start with a separatrix region in the $J_1 - \theta_1$ plane. For $N = 2$, J_2 is fixed by J_1 and energy conservation. For $N = 3$, on the other hand, energy conservation only defines a relationship between J_2 and J_3 allowing them to change in concert. We can visualize this new freedom by extending the J_2 axis out of our picture of the $J_1 - \theta_1$ plane and allowing the exact shape of the separatrix region to vary with J_2 . The essence of Arnold diffusion is the slow diffusion along the J_2 axis in this resonance layer. As J_2 and J_3 slowly change a different resonance in, say the $J_2 - \theta_2$ plane, could be reached. At this point the system dynamics can change to follow this new resonance while slowly diffusing along J_1 and J_3 . Because diffusion along resonance layers is so slow, for suitably short times the system dynamics can behave similarly to a corresponding two degree of freedom system.

Asymmetric Optical Billiards

Two dimensional billiards have proven to be very successful at explaining and predicting optical properties of ARCs. Even in cases where solution of the wave equation is possible the geometrical optics picture often provides an intuitive understanding which may be lacking in the wave solution. There are two main aspects of optical behavior of

ARCs which the ray model can address. These are the Q-factor (or photon storage time) and the emission directionality. This is quite good considering there is little else which is experimentally measurable besides resonant frequencies which are clearly outside the realm of geometrical optics. The fact that both of the properties that the ray model can address are related to escape from the cavity indicates that we must modify our picture of a hard walled boundary slightly.

The fundamental difference between an optical ARC and the billiards discussed thus far is that an ARC is an open cavity meaning that light may eventually escape. In the geometrical optics limit we assume that light undergoes total internal reflection at every reflection unless the angle of incidence is less than the critical angle. In the simplest model of a dielectric cavity one can make the approximation that at the critical angle the reflectivity of the cavity walls goes from one to zero. This is adequate for providing a qualitative understanding of the optical properties of ARCs in many cases. A more accurate model uses Fresnel's formulas to determine the probability of transmission through the cavity wall. Further improvements can be made by making corrections to Fresnel's formulas for reflection at a curved interface⁴⁹.

The method for determining the Q-factor of a mode in an ARC is quite straight forward. One simply calculates trajectories for an ensemble of rays with initial conditions corresponding to the mode of interest. Calculating the average path length L that the rays travel prior to reaching $\sin \chi = \sin \chi_c$ provides a measure of the Q-factor

$$Q = nkL . \quad (1.62)$$

It should be immediately apparent that in the classical limit modes corresponding to an initial condition $\sin \chi_0 > \sin \chi_c$ will have infinite Q for all deformations up to a threshold deformation ϵ_c which breaks the last KAM curve standing between $\sin \chi_0$ and $\sin \chi_c$.

The value of ϵ_c is clearly sensitive to the particular values for $\sin \chi_0$ and $\sin \chi_c$.

Performing such simulations with $\sin \chi_0 = 0.8$ and $\sin \chi_c = 0.5$, it has been found

$\epsilon_c \approx 0.07$ ³⁶. They also determined that in general the Q decreased with increasing deformation as

$$Q \propto (\epsilon - \epsilon_c)^{-\alpha} \quad (1.63)$$

where $2.5 < \alpha < 3.5$. The fact that α varies over only a small range for all $\sin \chi_0$ and

$\sin \chi_c$ results from the fact that the rate limiting step for transport to smaller $\sin \chi$ is always passage through the cantorus left behind by the breaking of the last KAM curve.

When the Q factor predicted by the ray model is larger than that experimentally observed in a dielectric resonator then evanescent (tunneling) escape will dominate the lifetime of a mode. Once the threshold deformation is crossed, however, classical escape quickly becomes the dominant escape mechanism.

It makes intuitive sense that breaking the rotational symmetry of a resonator should also force the emission from the resonator to be non-isotropic. If emission is limited to tunneling escape by the presence of KAM curves at large values of $\sin \chi$, however, the perturbation to the emission pattern might be much smaller than that of the shape since these KAM curves are only slightly perturbed from those of the circular cavity. Above the

critical deformation, when ray escape becomes possible, one would expect the emission to become more directional due to refractive escape at points of high curvature. As we have seen in every SOS the angle of incidence tends to be smallest for most trajectories at high curvature points. Although this argument predicts that the emission would be somewhat localized in the near field, the far field emission could be much less directional if refractive escape occurs with a wide range of angles of incidence.

A full ray simulation actually indicates that once refractive escape becomes possible the emission is highly localized in the near field and highly directional in the far field. Furthermore, simulations demonstrate that for a particular value of $\sin \chi_c$ the emission pattern is largely insensitive to the initial launching conditions. These remarkable properties can be understood by recalling that trajectories tend to follow adiabatic invariant curves as they diffuse through chaotic regions of phase space. Because the motion of a trajectory towards lower ICs is slow compared with the motion along an IC, refractive escape will occur when the ray reaches the IC which is tangent to a line drawn at $\sin \chi = \sin \chi_c$. Figure 13a shows the SOS of a quadrupole ($\epsilon = 0.7$) with the IC of a corresponding ellipse overlayed. For a cavity with index of refraction $n = 2$ this IC is tangent to $\sin \chi = \sin \chi_c = 0.5$. The arrows on the IC indicate the direction of the fast motion of the trajectory under the mapping T_ϵ^4 , chosen due to the proximity of the four bounce periodic trajectories. This fast flow along the IC ensures that the ray crosses the critical line for escape very near the point of tangency before the trajectory diffuses to a lower IC. This argument reveals that the refractive escape is localized both in ϕ and $\sin \chi$

which produces highly directional emission. The only condition placed on $\sin \chi_0$ is that it not be separated from $\sin \chi_c$ by a KAM curve. Thus we see that emission will occur tangentially from the points $\phi = 0, \pi$ for all refractively escaping modes.

As we have seen previously, the adiabatic invariant curve model fails when the IC intersects a stable island. Although island chains tend to fall along an IC a chaotic trajectory is not permitted to enter the islands and is instead forced to flow around them. This can lead to a different emission pattern when the line $\sin \chi = \sin \chi_c$ intersects a major island. Figure 13b depicts the same quadrupole shape as fig. 13a but with index of refraction $n = 1.45$. In this case, the point of tangency between the IC and $\sin \chi = \sin \chi_c = 0.69$ is positioned inside the stable four bounce island. Rather than emission occurring at this location, a ray will escape as it circulates around the island. Trajectories flow around the islands in a clockwise fashion following the stable and unstable manifolds. This flow pattern leads to refractive escape occurring beside the islands near $\phi = 3\pi/4$, and $7\pi/4$. As before, this emission pattern is insensitive to the initial condition and depends solely on the presence of a stable island at the point where the IC is tangent to the critical line. This displacement of the emission pattern by stable islands is called dynamical eclipsing. Dynamical eclipsing is an excellent example of a phenomenon which appears in wave solutions but is much more clearly understood through the ray model.

Two dimensional billiard models have been used to explain the emission properties of a number of dielectric cavity based lasers. In highly asymmetric quadrupolar

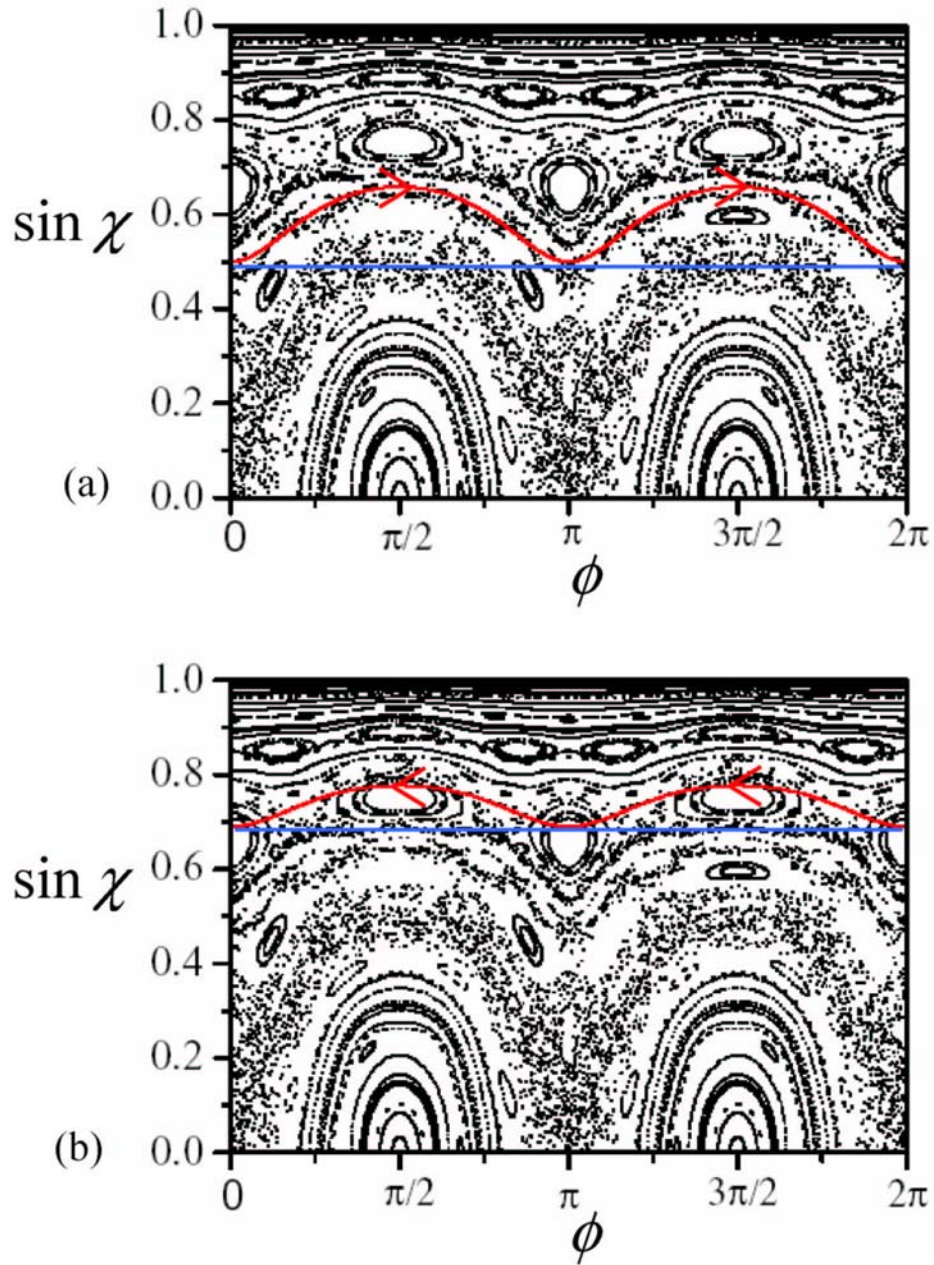


FIGURE 13. Refractive escape from an asymmetric resonator occurs near the point where an adiabatic invariant curve (red) is tangent to the line $\sin \chi = \sin \chi_c$ (blue). (a) For a resonator with index of refraction $n = 2$ this happens at the points of highest curvature. (b) For a fused silica resonator with $n = 1.45$ a stable island prevents trajectories from reach the expected escape point. Dynamical eclipsing causes refractive escape to occur to the small ϕ side of the stable island.

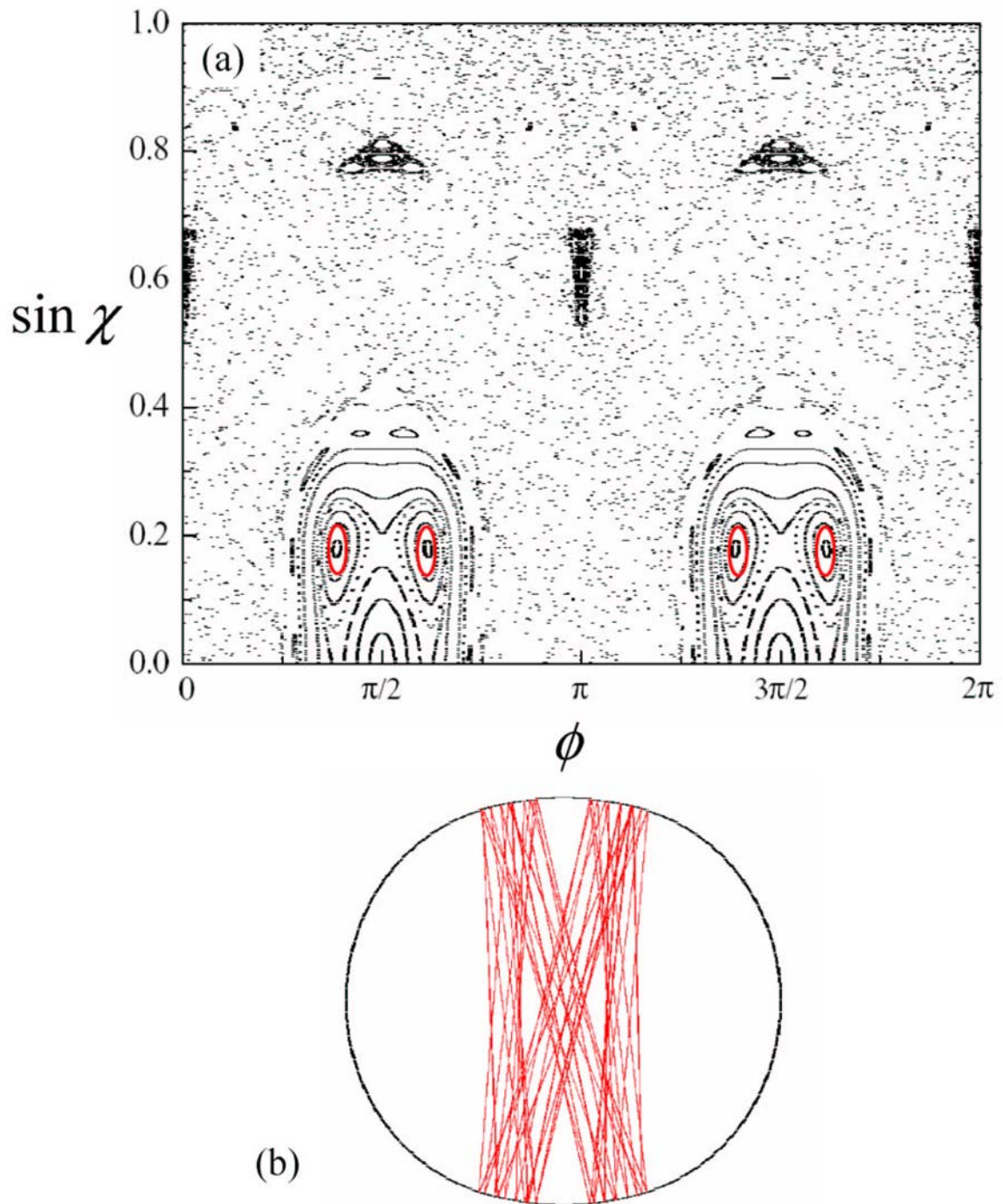


FIGURE 14. Bowtie modes can be supported in high index semiconductor microdisk lasers. (a) Poincaré surface of section for quadrupole ($\epsilon = 0.125$). Stable bifurcation of two bounce orbit is indicated in red. (b) Real space trajectory of a quasiperiodic orbit near indicated location in (a).

semiconductor laser cavities yet a different type of refractive emission pattern is observed²³. Figure 14a shows that for a quadrupolar deformation $\epsilon \approx 0.125$ a strong secondary resonance exists around the stable two bounce orbit. Modes associated with this stable orbit are referred to as bow-tie modes due to their real space trajectory (fig. 14b). Bow tie modes can have a moderately high Q-factor in a semiconductor cavity ($n \approx 3.3$) enabling high power, highly directional laser emission. Emission properties of lasing droplets have also been explained using a modified two dimensional billiard model²⁰. Because the droplets possess azimuthal symmetry the three dimensional Hamiltonian is partially separable. The component of angular momentum along the symmetry axis L_z is conserved. This conservation allows the dynamics to be reduced to an effective two dimensional billiard where the effect of the motion around the symmetry axis causes rays to move on parabolic curves between reflections. This modified two dimensional billiard has been called a centrifugal billiard because the motion perpendicular to the symmetry axis can be treated as resulting from a centrifugal force. It should be noted that although the experimental system is three dimensional, the conservation of L_z leaves only two degrees of freedom. As such, Arnold diffusion does not occur in this system.

CHAPTER IV

EXPERIMENTAL TECHNIQUES

As discussed in the Introduction fused-silica microspheres have been studied by a number of research groups. As such, a variety of techniques for the fabrication of glass microspheres and the characterization of WG modes have been described in the literature^{50,51}. The beginning of this chapter will present the details of the fabrication technique that we have developed during the course of our studies of microspheres. In addition, we will describe how strongly deformed microspheres have been produced for the first time. A few common techniques for launching WG modes in glass microspheres will be discussed followed by a detailed description of the prism coupling method, which is employed for the majority of experiments presented in this dissertation. Techniques for characterizing the resonance properties of microspheres will be presented and specific features of WG mode spectra will be demonstrated with experimental data. The chapter concludes with a description of the complete experimental setup employed to study the optical properties of deformed fused-silica microspheres.

Fabrication of Fused-Silica Microspheres

Fused silica microspheres are formed from standard optical fiber by heating with a CO₂ laser. A convenient choice of fiber is the TECSTTM hardclad multimode fiber

produced by 3M. The polymer cladding is easily removed with acetone or by heating with the laser. This ensures that the material from which the microsphere is formed is pure fused silica with a uniform index of refraction ($n = 1.457$). It is easier to make spheres with a diameter smaller than that of the fiber core so a good choice is to use fiber with a 200 μm core diameter for maximum flexibility.

To make a microsphere, a length of fiber is cut to approximately 2 inches and mounted in a standard fiber chuck. A small weight (750 mg) is taped to the free end of the fiber. The fiber chuck is mounted vertically on a 3 dimensional translation stage with the weighted fiber hanging below it. A 10W CO₂ laser is focused by a ZnSe lens to a point on the fiber midway between the weight and fiber chuck. Alignment of the system is aided by an imaging system consisting of a microscope objective and CCD camera with telephoto lens oriented perpendicularly to the laser beam. The camera with telephoto lens requires an object distance of at least a few meters. This condition is achieved by placing an objective lens slightly less than one focal length away from the fiber such that a virtual image of the fiber is produced far in front of the CCD camera. This system provides a compact video-microscope with easily adjustable magnification. Once the focal plane of the imaging system is made to contain the focal point of the laser it becomes quite simple to repeatedly align fibers with the laser focus. The imaging system also provides essential visual feedback during the fabrication of the microsphere.

Once the fiber is aligned in the focused beam the buffer and hard cladding are burned off with the laser set to a low power, just short of what would cause the fiber to begin stretching under the tension provided by the hanging weight. Once a suitable

section of the fiber is cleared of the cladding the laser power is increased causing the fiber to stretch into a narrow stem. If the microsphere is intended to be placed near the surface of a prism for input coupling to WG modes the stem must have a smaller diameter than the sphere for a length greater than the distance from the edge of the prism to where the sphere will be positioned. Typically this distance is on the order of a few millimeters. While it is possible to form long stems which are only a few microns thick it is often best to make the diameter of the stem about one quarter to one third that of the sphere in order to achieve maximum stiffness and minimize vibration amplitudes.

When a stem of suitable dimensions is achieved the attached weight and excess fiber must be removed. It is helpful to leave a lump of material at the base of the stem from which to form the microsphere. Directly below this lump the fiber is heated and stretched until it breaks leaving the stem with an unshaped quantity of glass. To this point the weight has served the dual purpose of providing the tension necessary to stretch the fiber and also to maintain the straightness of it. Once the weight is removed, the optical fiber will tend to bend towards the side that is being heated by the laser. This occurs in response to unbalanced surface tension forces on the hot and cold sides of the stem. As the lump of glass at the end of the stem is heated, surface tension forces cause it to form into a nearly perfect sphere. It is possible to increase the size of the sphere by melting more of the stem into it. It is also possible to reduce the size of the sphere by increasing the laser power causing sublimation of the glass. This is not recommended, as it seems to adversely affect the quality of the microsphere, perhaps by microcrystallization. As the sphere forms, it tends to bend towards the laser as previously

described. In order to produce a rotationally symmetric spheroid the fiber must then be rotated 180° so that the sphere can be bent back into line by gentle heating with the laser. The result is a slightly prolate fused silica microsphere which is perturbed by hanging from the stem under its own weight. The microsphere retains circular cross sections in the planes perpendicular to the stem. The largest circular cross section is often referred to as the equator and the plane containing it as the equatorial plane. The slight ellipticity of the microsphere causes modes which are inclined to the equatorial plane to precess and acquire a frequency red-shift as described previously.

Microspheres ranging in diameter from less than 10 μm to about 200 μm can be fabricated using this technique. The lower size limit is due more to approaching the diffraction limit of white light imaging rather than material properties. Much larger spheres can also be made using larger diameter fiber or glass rods. One limitation of this technique is that it does not translate well when working with glasses other than fused silica. Doped silica and phosphate glasses typically have much sharper melting transitions which require very precise control of the laser power within a rapid response time. In addition, these glasses do not have the remarkably low thermal expansion of fused silica. This results in much more surface roughness upon cooling. Finally, the doped materials are typically even more prone to microcrystallization than pure fused-silica.

Fabrication of Deformed Fused-Silica Microspheres

The primary challenge faced in producing a deformed fused silica microsphere is that molten glass tends to take the lowest energy shape of a sphere. Although perhaps possible, it is not practical to deform a molten sphere by physical contact or applied electric fields. Mechanical deformation of fused silica microspheres by squeezing and stretching has been demonstrated for WG mode tuning, however, these deformations remain quite small and don't deform the circular equatorial cross section of the sphere. The solution developed for the experiments presented in this dissertation is inspired by the asymmetric liquid jet produced through a rectangular orifice. The fact that surface tension tends to form a high symmetry shape can be exploited by starting with an appropriate strongly asymmetric shape. Based on our experience with fused silica microspheres we have found that a convenient strongly asymmetric starting shape can be formed by placing two similar sized microspheres in contact. As the two spheres are heated and fuse together, the shape will tend towards the familiar shape of a standard microsphere. Before reaching the final spherical shape, however, this object will pass through a continuum of asymmetric shapes. At a certain stage in this evolution the glass will become completely convex, but with a large aspect ratio. The intensity of the CO₂ laser which provides the heating can be controlled such that this shape relaxation occurs on a convenient time scale. When a desired shape is reached the laser can be switched off, freezing the glass in an asymmetric shape.

The procedure we have developed for producing high quality deformed fused silica microspheres is as follows. First, two microspheres of the same size are produced as previously described. The first microsphere is formed at the end of a very thin stem. The second is formed with a more substantial stem, which is to be the stem that will remain attached to the final deformed microsphere. The center of the finished deformed microsphere is midway between the centers of the two original spheres. Therefore, it is best if the second microsphere is allowed to bend considerably on its stem in order that when the spheres are placed in contact the stem can be closer to the center point. The second sphere is left mounted to the 3D translation stage and positioned at focal point of the CO₂ laser. The first sphere is then mounted from below and positioned in contact with the second. The line containing the centers of the two spheres should be perpendicular to the two stems which are parallel. In this configuration the spheres are heated slightly with the laser until fusion occurs at the contact point. Once the two spheres are welded together the thin stem can be broken off or melted through. The double sphere can then be heated again allowing the two spheres to melt into one another until the surface tension produces a completely convex surface (no waist at the joint). Once a convex shape is obtained the object can be considered as a deformed microsphere and is suitable for resonance studies. The eccentricity of the microsphere can be continually reduced in small amounts by gently heating it with the laser. As such, a single microsphere can be studied over a wide range of

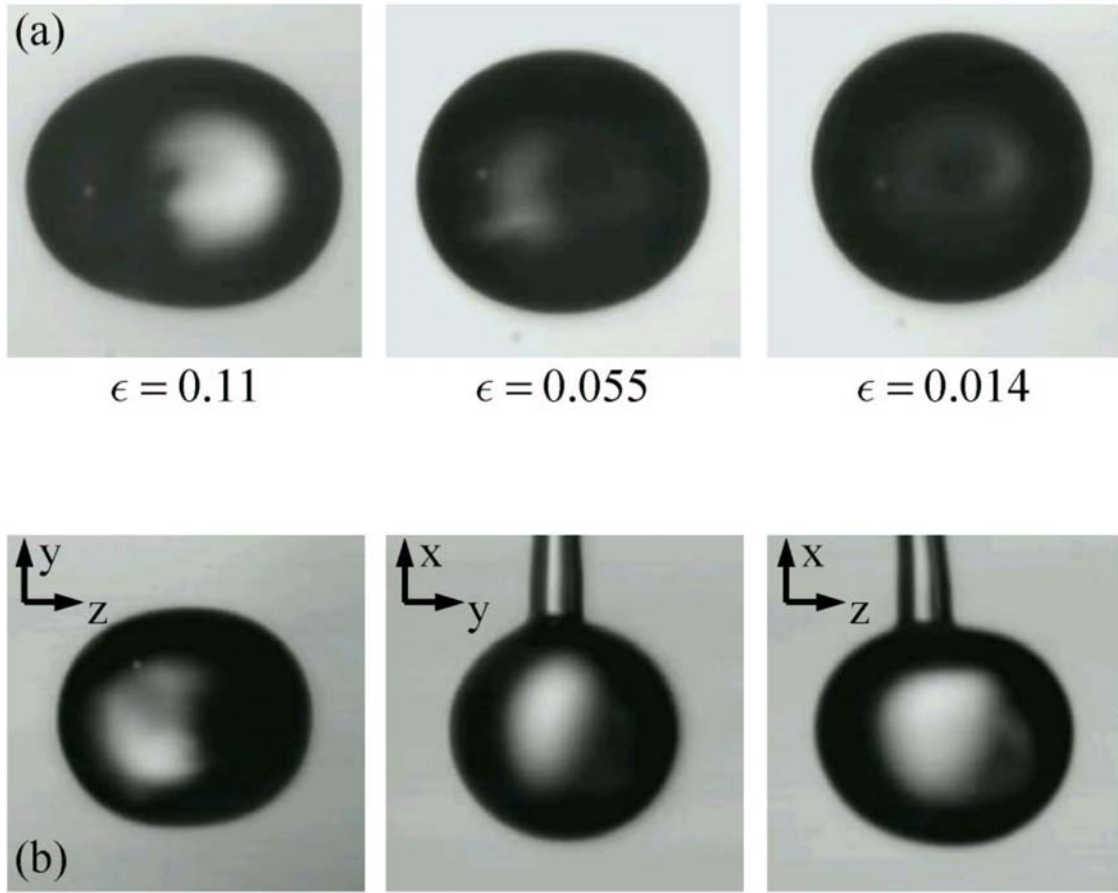


FIGURE 15. Deformed fused-silica microspheres. (a) Sequence of images of a single deformed fused-silica microsphere at different deformations. (b) Three orthogonal perspectives of a deformed fused-silica microsphere. The z -axis is along the line connecting the centers of the two original spheres. The x -axis is defined by the stem. The stem breaks the symmetry about the z -axis.

eccentricities. Figure 15a shows a series of images of a single deformed microsphere as its eccentricity was repeatedly reduced. The eccentricity

$$\epsilon = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} \quad (1.64)$$

is defined to agree with the deformation parameter of the quadrupole.

The shape of the deformed microsphere that results from this procedure can be thought of as a sphere which has been stretched along an axis which is perpendicular to the stem. Because this long axis is an approximate symmetry axis we define it to be the z axis of the resonator and the stem is chosen to be the x axis. This is in contrast to a standard microsphere where the symmetry axis is along the stem. Figure 15b shows three orthogonal views of a strongly deformed microsphere. The nearly circular projection onto the x-y plane shown in the central frame confirms that the microsphere has approximate rotational symmetry about the z axis which is broken by the presence of the stem. The slow relaxation process allows high order fluctuations in the surface shape to completely disappear and the resulting shapes are well approximated by the lowest order (dipole and quadrupole) terms.

Launching Whispering Gallery Modes

One property of WG mode resonators that sets them apart from standard Fabry-Perot cavities is that the coupling efficiency and Q factor are not intrinsic to the cavity itself. In a Fabry-Perot cavity these properties are fixed by the reflectivities of the constituent mirrors. In a microsphere resonator, on the other hand, the coupling efficiency and effective Q factor are controllable through a variety of coupling techniques. If we rewrite equation (2.22) as

$$Q^{-1} = Q_0^{-1} + Q_c^{-1} \quad (1.65)$$

where Q_0^{-1} represents all losses not associated with the coupler, then we can see that the net Q of the cavity can be dominated by either the bare cavity Q_0 or by output through the coupler. The amplitude of the WG mode field will be maximum when the coupling rate is equal to the uncoupled loss rate, $Q_c^{-1} = Q_0^{-1}$. In this critical coupling case the net Q of the coupled cavity is $\frac{1}{2}Q_0$. In the overcoupled regime, $Q_c^{-1} > Q_0^{-1}$, output through the coupler is the dominant loss mechanism for the WG mode, causing a reduced Q factor and preventing the field from building to its maximum amplitude. In the undercoupled regime, $Q_c^{-1} < Q_0^{-1}$, the Q factor can be made to approach Q_0 , however the mode field amplitude is then limited by the small input coupling rate. It is worth noting that for microsphere applications where one wishes to use a significant fraction of the coupled light, the cavity must be operated in the overcoupled regime since all other radiative losses, whether by scattering or evanescent leakage, are isotropic.

There are a number of techniques for coupling light into the high Q WG modes of a microsphere¹⁴. With any technique there are two main considerations that determine the efficiency of exciting a particular mode. First, the exciting laser wavelength must be resonant with the mode. For all of the techniques described this is accomplished by tuning the wavelength of the laser across the resonance. The second consideration is mode matching between the laser and the WG mode. The symmetry of geometrical optics requires that the input light must correspond to the time reversal of light that is emitted from the mode.

One way to view the mode matching condition is to recognize that there are infinitely many wave vectors, $\mathbf{k}$, which correspond to the resonant wavelength, ($\lambda = 2\pi/|\mathbf{k}|$) but not all of them correspond to wave vectors in the WG mode. WG modes correspond to ray trajectories which are bounded by repeated total internal reflection at glancing incidence. The transmitted wave vector is parallel to the tangent at the point of reflection. Therefore, any successful input coupling technique must produce a substantial field with wave vectors parallel to a tangent of the sphere surface.

As discussed earlier, only light passing the sphere with impact parameter $a < b < na$ can satisfy conservation of angular momentum when tunneling into the sphere to become trapped by total internal reflection. Another way to view this process is by diffraction. A ray of light passing near the sphere can be diffracted into a trapped trajectory inside the sphere. This type of process corresponds to the exact time reversal of emission from the sphere. Based on this observation one might suggest using a simple free-space method of launching WG modes where a laser beam is focused at the edge of the sphere. The primary advantage of this approach is that it does not require precise positioning of a physical object in close proximity to the microsphere. In addition to the practical advantage of positioning a focal spot rather than a physical object this technique causes no perturbation to the WG modes and therefore does not degrade the Q factor of the modes as other techniques do. Although useful in some situations this technique has several drawbacks. First, the input coupling efficiency to a specific mode is determined entirely by the Q factor of the mode. Therefore, the coupling efficiency is extremely low

for high Q modes. Furthermore, this means that the technique couples preferentially into low Q , large ν modes rather than the desired high Q , $\nu = 1$ modes. This undesirable bias can also be seen by the fact that the angle of incidence for a ray inside the sphere is determined by its impact parameter. Larger impact parameters correspond to larger angles of incidence and lower ν modes. Unfortunately, larger impact parameters also correspond to smaller coupling efficiencies due to the greater tunneling distance. A final technical disadvantage of this technique is that it often results in a large amount of uncoupled scattered light which can make observation of the WG modes difficult.

An excellent near field coupling technique for launching WG modes is to use a modified single mode optical fiber. Launching WG modes by optical fiber is very attractive for a variety of experimental and practical applications where optical fiber provides a convenient means of manipulating light. One way to do this is by polishing through the cladding of a curved optical fiber to expose the core⁵². This allows a strong evanescent field to be accessible to the microsphere for excitation of the WG modes. A similar technique requires heating and stretching a single mode fiber to produce a tapered region with a diameter of a few microns⁵³. Similarly to the polished fiber technique, this makes a strong evanescent field accessible to the microsphere. In both cases low ν modes can be selectively excited by matching the index of refraction of the fiber to that of the sphere. The fiber taper technique can achieve even better mode matching to WG modes by choosing the ideal diameter of the fiber taper. Greater than 99.8% power transfer from a fiber to a WG mode has been demonstrated for a dual-tapered-fiber coupling device⁵⁴. This remarkable efficiency does come at the price of reduced Q and

increased technical difficulty. The resulting Q factor for the most efficiently coupled mode reported was approximately 10^6 . Increased Q factors can be achieved with smaller coupling efficiencies if the fiber taper is not in contact with the microsphere. With this configuration, however, it is extremely difficult to maintain vibrational stability. Fiber coupling techniques also suffer from limitations such as being restricted to wavelengths for which the fiber is single mode. Also efficient coupling to high Q modes can only be achieved for a certain range of sphere sizes due to practical limits on the taper diameter.

The most versatile near field coupling device is a high index prism. Similarly to the fiber coupling techniques, WG modes are launched by positioning the sphere in a strong evanescent field. With a prism coupler the evanescent field is produced by focusing a laser beam to a spot on an inside face of the prism at an angle of incidence near the critical angle for total internal reflection. Excitation of WG modes can be achieved by frustrated total internal reflection when the microsphere is positioned near the laser focus spot with a small gap between the sphere surface and prism face. Mode matching to any specific mode can be achieved by adjusting the angle of incidence, Φ , of the input beam inside the prism and adjusting the angular size of the beam, $\Delta\Phi$ and $\Delta\Theta$.

For $\nu = 1$ modes

$$\sin \chi = \frac{l}{n_s k a} \approx 1 \quad (1.66)$$

where n_s is the index of refraction of the sphere. Applying Snell's law we can see that inside the prism (index n_p)

$$\sin \Phi = \frac{l}{n_p ka} \approx \frac{n_s}{n_p}. \quad (1.67)$$

This formula demonstrates the importance of using a high index prism in order to make the angle of incidence inside the prism conveniently smaller than 90° . If one wishes to excite $\nu > 1$ modes Φ is simply decreased appropriately. It can also be shown that the angular size of the far field emission from a low ν , $|m| = l$ WG mode through a prism coupler is approximately Gaussian with

$$\Delta\Phi^2 = \frac{\sqrt{n_s^2 - 1}}{n_p^2 ka \cos^2 \Phi} \quad \text{and} \quad \Delta\Theta^2 = \frac{n_s + \sqrt{n_s^2 - 1}}{n_p^2 ka}. \quad (1.68)$$

A laser beam focused on the prism face and matching conditions (1.67) and (1.68) will provide optimal mode matching to the fundamental WG mode in the equatorial plane of the sphere. Coupling efficiency of about 30% can be achieved with a prism coupler and standard objective lens. Using cylindrical focusing optics, prism coupling efficiency as high as 80% has been demonstrated. In many cases coupling efficiency is not of primary interest and prism coupling is then often the best choice due to its flexibility. Excitation of $|m| \neq l$ WG modes can be achieved by choosing the plane of incidence of the focused laser beam to be inclined by an angle Θ relative to equatorial plane of the sphere. For this case, conditions (1.67) and (1.68) describing proper mode matching are generalized to³³

$$\sin \Phi = \frac{m}{n_p k a \cos \Theta} \quad \text{and} \quad \sin \Theta = \frac{\sqrt{l(l+1)-m^2}}{n_p k a} \quad (1.69)$$

and

$$\Delta \Phi^2 = \frac{\sqrt{n_s^2 - 1}}{n_p^2 k a \cos^2 \Phi \cos^2 \Theta} \quad \text{and} \quad \Delta \Theta^2 = \frac{\sqrt{n_s^2 - 1}}{n_p^2 k a \cos^2 \Theta}. \quad (1.70)$$

A single prism can, therefore, be used to couple into any WG mode at any wavelength by only adjusting the laser focusing optics. Unlike fiber couplers, the only sensitive degree of freedom for positioning the sphere and prism is the distance between them since positioning of the total internal reflection spot is again performed by adjusting the focusing optics. The amplitude of the evanescent field produced at the total internal reflection spot on the prism face decays exponentially with distance from the surface. As such the coupling rate between the WG mode and prism coupler depends exponentially on the sphere-prism gap. By precisely adjusting this gap using a piezoelectric translation stage, a WG mode can be studied across the full range coupling strength from overcoupled to undercoupled.

Characterizing Whispering Gallery Modes

There are two basic ways to measure the resonance properties of microsphere WG modes. First, one can measure the transmitted pump intensity. When the pump laser is resonant with a WG mode energy will be absorbed by the mode from the pump resulting in a decreased transmission intensity. In order for this technique to be practical one must

achieve large coupling efficiencies. With low coupling efficiencies the challenge lies in measuring small changes to a large amplitude signal. Although this technique is straightforward in principal, one technical challenge can be eliminating reemitted light from the measurement. The most common motivation for measuring transmission intensity is to quantify the coupling efficiency. The second approach to measuring WG modes is to collect only light which was coupled into the sphere. With this approach coupling efficiency is a secondary concern as long as the mode field has sufficient amplitude for detection. This is the approach employed throughout this dissertation.

In order to collect light from a WG mode one can collect light which is coupled back out of the mode through the coupling device. In a single fiber-taper configuration the output coupled light is indistinguishable from the transmitted pump beam. A second fiber taper can be used to collect only light coupled into the mode. It has also been demonstrated that WG modes can be mapped by using a micron sized optical fiber tip as a near field probe. With a prism coupler it is possible, under certain circumstances, to distinguish the WG mode emission pattern from the reflected pump beam. This is not completely satisfactory however because they are, by necessity, in nearly the same direction and therefore overlap considerably producing a large DC background signal. This situation can be remedied by launching WG modes which are inclined to the equator of the microsphere. As previously discussed, an inclined mode $lm\nu$ can be considered equivalent to a frequency shifted equatorial mode $l\nu$, which is inclined and precessing about the symmetry axis. Light in such a mode will couple out through the prism along the reflected pump beam but also along the direction symmetric about the equatorial

plane of the sphere. This second emission lobe can be measured with no background from the reflected pump. It is interesting to note that this geometry provides an often desirable condition where output coupling rate through the prism is decreased while the input coupling rate is unaffected. This occurs because as the inclined mode precesses it actually only visits the point of closest contact with the prism after an integer or half integer number of precessions. Unfortunately these precessing modes are unsuitable for some applications because the effective mode volume is considerably greater than that for the llv mode.

A second approach for measuring light from WG modes is to measure the far field emission directly from the microsphere. The primary disadvantage of this approach is that a microsphere radiates isotropically in the plane of the mode and it is only practical to collect a fraction of the emitted light. This technique is attractive, however, due to its simplicity and it can also provide efficiency comparable to or greater than the output coupled measurement when one is interested in measuring the ultimate cavity Q factor in the undercoupled regime. Also, additional information about a sphere's emission characteristics can be obtained by imaging the sphere.

The Q factor of a WG mode can be measured by directly observing the exponential decay of energy stored in the mode as described by equation (2.19). These ring-down measurements have been performed using an acousto-optic modulator to periodically switch the pump laser off allowing the mode field to decay. Light from the mode was collected into a photomultiplier tube and the time dependant signal was averaged and recorded using a digital oscilloscope. Q factors approaching 10^9 were

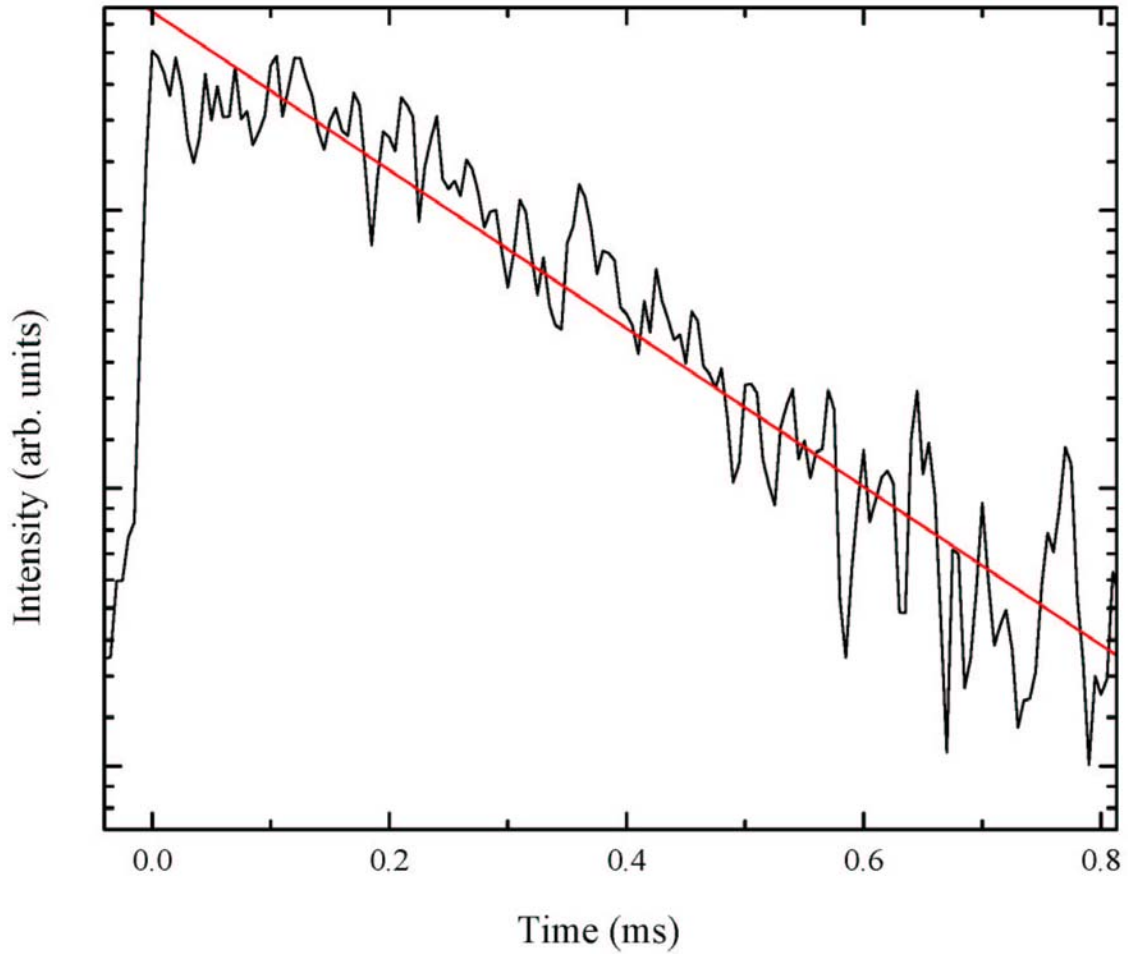


FIGURE 16. Cavity ring-down measurement of photon lifetime in a WG mode. The measured time constant for this mode is $\tau = 0.36 \mu\text{s}$ corresponding to 0.9×10^9

measured by fitting an exponential curve to the data as shown in figure 16. The major disadvantage of this technique is that study is limited to a single resonance at a time and overall spectral features cannot be determined effectively.

In order to characterize WG modes in a microsphere it is useful to record a resonance spectrum. This is accomplished by recording the intensity of light that is

emitted from the microsphere as the pump laser wavelength is scanned across successive resonances. The emitted radiation is measured by a photodiode using a lock-in detection technique where the pump laser is chopped at approximately 1 kHz. For the experiments presented in this dissertation the pump laser is a New Focus Velocity external cavity tunable diode laser operating near either $\lambda = 630 \text{ nm}$ or $\lambda = 800 \text{ nm}$. These lasers are capable of continuous tuning over a range of approximately 20 nm.

There are three modes of operation for controlling the laser wavelength. First, the laser wavelength can be scanned across the full wavelength range at speeds as slow as 0.01 nm/s . This is accomplished inside the laser by adjusting the external feedback cavity length using a DC motor. This coarse scanning function is operated either from the front panel of the laser controller or by computer interface and is useful for obtaining the large scale structure of the WG mode spectrum. The spectrum is obtained by recording the radiated intensity as fast as the data acquisition computer is able. This DC scan, however, is not smooth enough or slow enough to enable precise, high resolution measurements of high Q WG resonances. The second scanning mode tunes the laser wavelength using a piezoelectric transducer (PZT) while the DC motor position remains fixed. The voltage applied to the PZT can also be controlled either manually or by computer interface and allows a scan range of approximately 0.1 nm with $2.5 \times 10^{-5} \text{ nm}$ resolution. The third mode of operation allows this resolution to be improved even further by directly applying a voltage to the PZT using a stable, high precision, digital power supply. Q factors as high as 2×10^8 can be measured reliably using the external voltage source.

Whispering Gallery Mode Spectra

The resonance spectrum of a fused-silica microsphere provides an immense amount of information about its physical properties. It is quite challenging to precisely calculate the resonance frequencies for a real microsphere *a priori*. In fact, small uncertainties in the exact physical characteristics of the microsphere can lead to very large uncertainty in calculated resonances. This sensitivity can be exploited, however, by measuring resonance frequencies and fitting the spectrum to a calculated spectrum. Such a procedure can lead to very precise determination of the microsphere radius, ellipticity and index of refraction. Fortunately, the spheres radius and ellipticity can even be determined quite accurately without requiring a full fit to a calculated spectrum. This is particularly convenient since the index of refraction is typically already well known by the choice of material. In order to exploit the sensitive measurement capabilities of microspheres one must understand the signatures of various physical effects on the WG mode spectrum. A detailed understanding of WG mode spectra is also required for many microsphere applications. For microsphere devices such as filters, microlasers, and laser stabilizers, it is necessary to manipulate the physical properties of the microsphere in order to provide the optimal mode spacing, frequency stability, and coupling strength to achieve the desired performance.

The dominant structure in a WG mode spectrum is the free spectral range. A microsphere acts similarly to a Fabry-Perot cavity of length $L = \pi na$ such that the free spectral range is

$$\Delta \nu_{fsr} = \frac{c}{2L} = \frac{c}{2\pi na}. \quad (1.71)$$

This is exactly the same as (2.16) which was derived by considering the change in frequency resulting from $\Delta l = 1$. The low resolution spectrum in figure 17a clearly shows the resonance pattern which repeats with a period of 1.2 nm. For 800 nm light this indicates that the radius of the sphere is 59 μm . This is in good agreement with the measurement of the diameter performed with an optical microscope. Figure 17a also shows the frequency shift between TE (black) and TM (red) modes. The different phase shifts acquired by TE and TM modes upon total internal reflection lead to TE modes being red-shifted from the TM modes with the same mode indices by an amount

$$\Delta \nu_{pol} \simeq \frac{\sqrt{n^2 - 1}}{n} \Delta \nu_{fsr} \quad (1.72)$$

for large l modes⁵⁵.

The resonance line shapes in figure 17a are not resolved due to the low resolution provided by the DC motor scan. In addition, the microsphere is in contact with the prism making it strongly overcoupled, which results in severely broadened resonances. The broad widths of the spectral features of fig. 17a actually represent the envelope of modes excited with different m which are not individually resolved. The multiple peaks within one free spectral range represent sets of modes with the same l but different ν . Modes

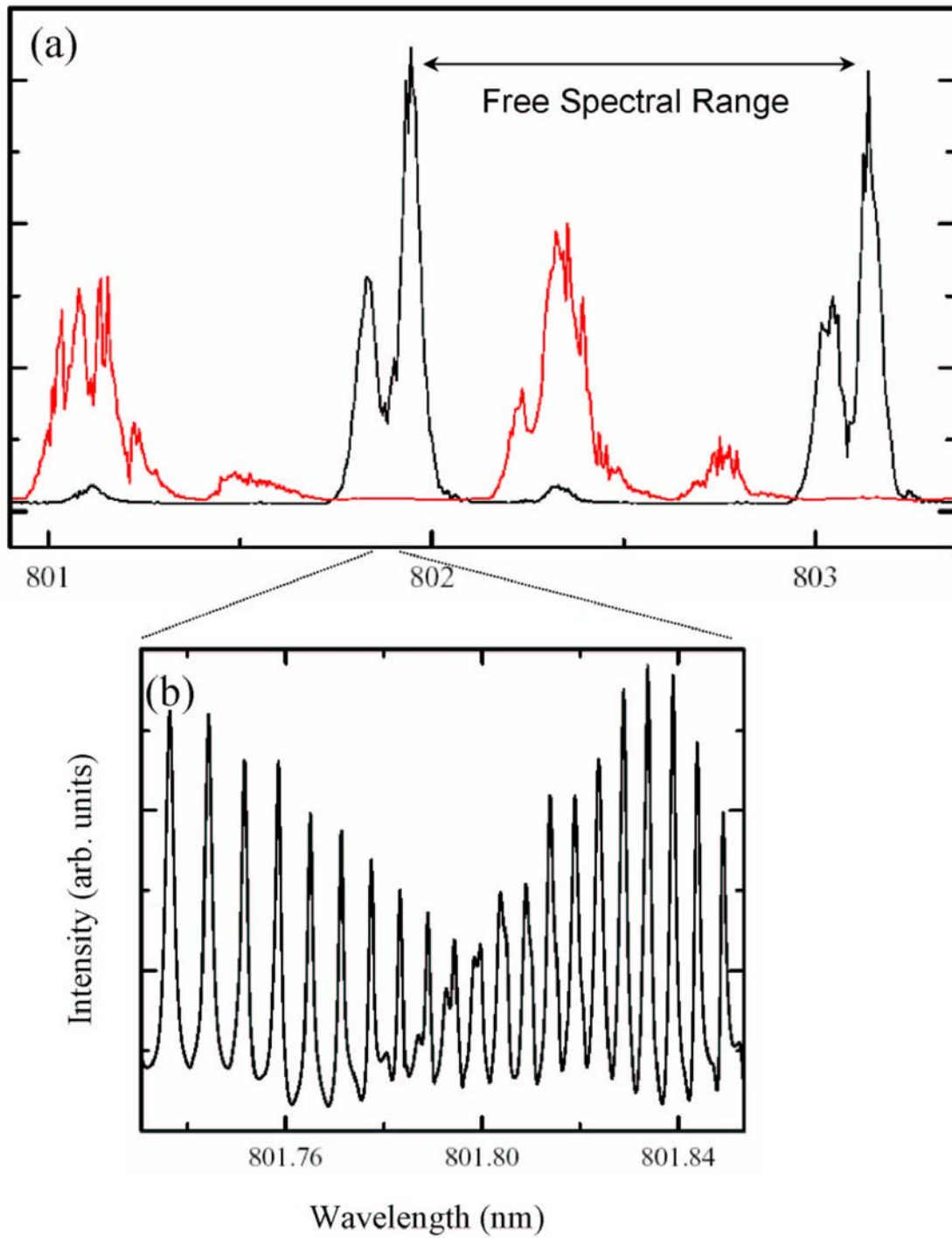


FIGURE 17. WG mode spectrum. (a) Mode structure repeats every free spectral range. TE modes (black) are red-shifted relative to TM modes (red). (b) Ellipticity of the microsphere causes splitting of different m modes.

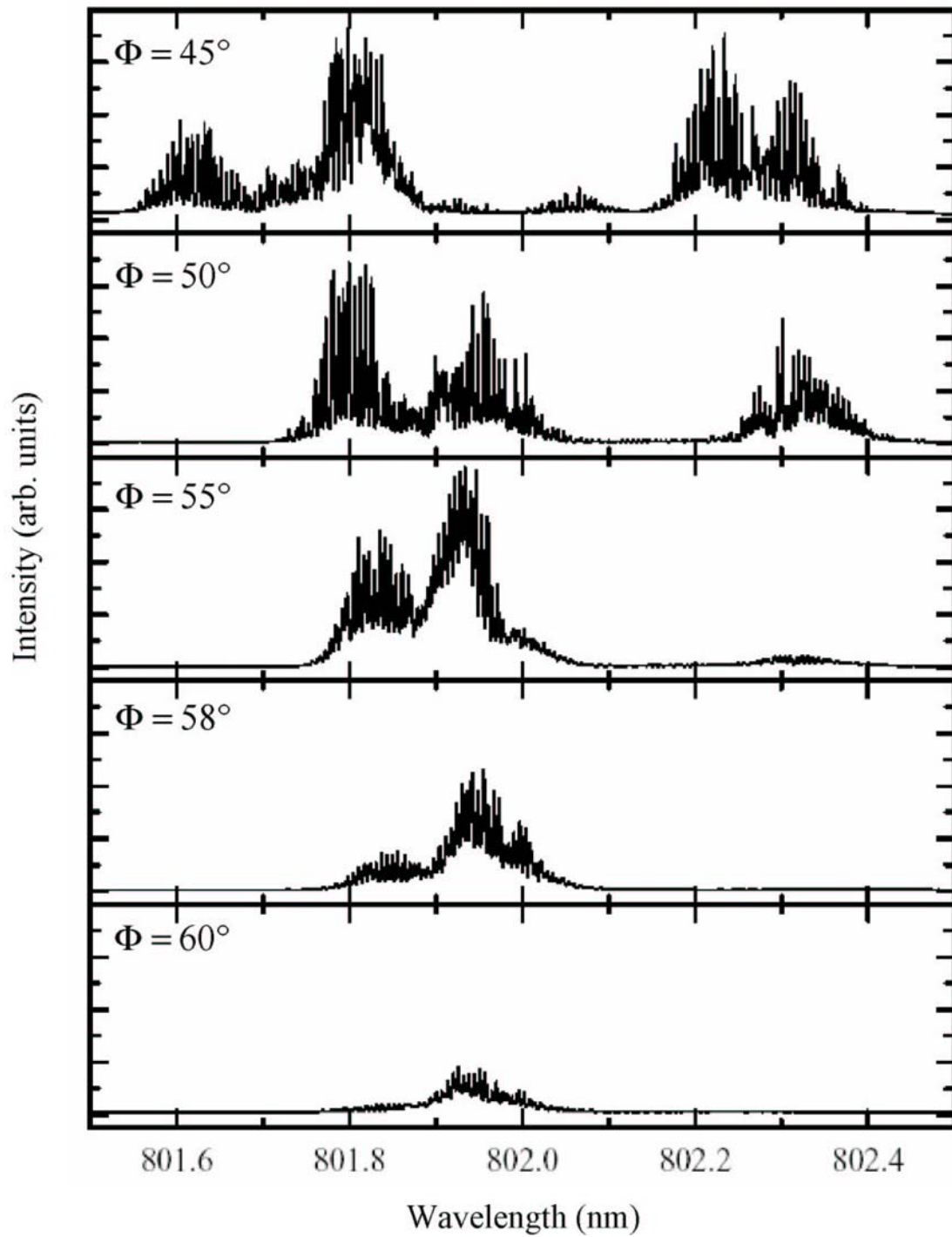


FIGURE 18. Dependence of WG mode spectrum on input angle. As the angle of incidence inside the prism Φ is decreased, lower ν modes are preferentially excited. The critical angle for total internal reflection between the prism and sphere is 58° .

with different ν are excited due to the range $\Delta\Phi$ of input angles inside the sphere.

Figure 18 demonstrates how increasing the angle of incidence Φ inside the prism leads to excitation of lower ν modes. When Φ approaches the critical angle for total internal reflection between the prism and sphere (1.67) $\nu = 1$ modes are preferentially excited. In the case shown $\nu > 1$ modes are still excited because $\Delta\Phi$ is greater than the optimal mode matching value specified by (1.68).

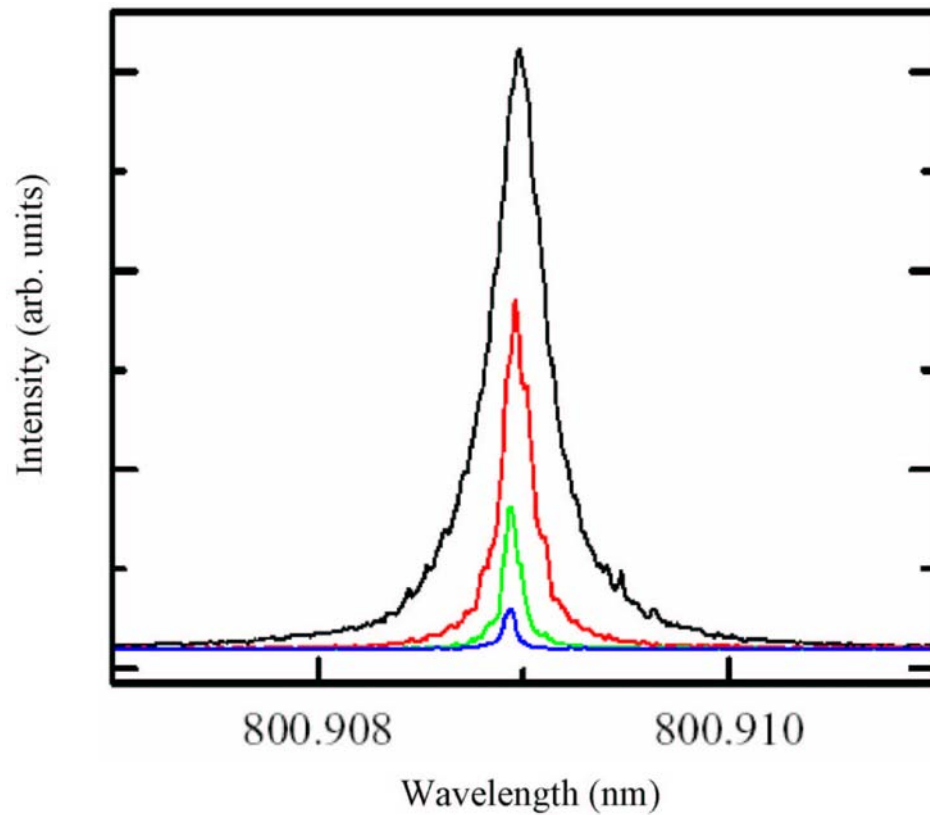


FIGURE 19. Q-factor increases with decreased coupling to prism. As the sphere is moved away from the prism the linewidth decreases due to decreased output coupling and the radiated intensity decreases due to decreased input coupling.

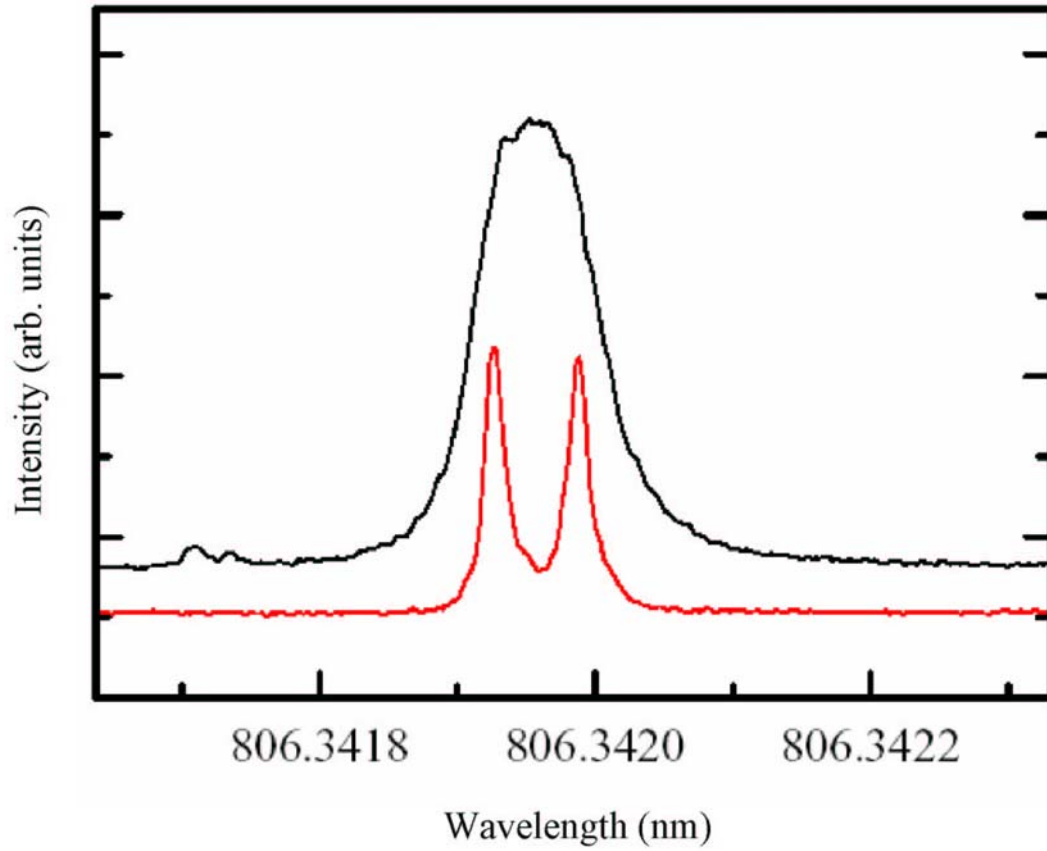


FIGURE 20. Mode splitting due to backscattering. Backscattering breaks the degeneracy between CW and CCW modes producing a doublet which is only observable for high-Q modes.

With the sphere pulled slightly away from the prism the fine structure of the spectrum due to the different m modes can be seen as in figure 17b. The splitting between the $m \neq l$ modes in this case is $\Delta\lambda = 0.005$ nm, which by equation (2.18) implies that the sphere has an ellipticity of 0.07%. The different spacing for different ν modes is also apparent. The Q-factor of the modes in figure 17b is less than 10^6 and is still dominated by output coupling through the prism. As the sphere is pulled further

from the prism, the input and output coupling rates through the prism decrease and the Q-factor increases as shown in figure 19. As the Q increases further the mode is sometimes observed to split into a doublet as shown in figure 20. This splitting is the result of backscattering which breaks the degeneracy between $\pm m$ (CW and CCW) modes. The splitting is an intrinsic property of the modes and is not power dependent. The magnitude of the splitting has been well explained by the backscattering from the same Rayleigh scatterers that contribute to the finite Q limit imposed by fused-silica⁵⁶. These scatterers produce a coupling between $+m$ and $-m$ modes with a coupling rate

$$\Omega = \frac{\omega_0}{2} \left(\frac{\alpha^2 \rho_{sc}}{V_m} \right)^{\frac{1}{2}} \quad (1.73)$$

where V_m is the effective mode volume, α is the average linear polarizability of the Rayleigh scatterers and ρ_{sc} is their number density. When this coupling is accounted for in solving Maxwell's equations the new eigenmodes are superpositions of the $\pm m$ modes.

Alternatively, one can consider the doublet of experimentally excited CW modes as the symmetric and antisymmetric superpositions of the new eigenmodes. Using this model, experimentally observed splitting indicates that the back scattered intensity is approximately 10^{-10} per round trip. This leads to a very small, but measurable, amount of backscattered radiation which has been exploited for diode laser stabilization by external optical feedback⁵⁷. The splitting is only observable when $Q > \omega_0/\Omega$ which also corresponds to the condition required for a significant field to build up in the CCW direction.

For some applications, such as cavity QED experiments, it is desirable to have an extremely small mode volume. The effective mode volume for WG modes is much less than the volume of the sphere that contains them. This is because the mode is confined near the surface of the sphere and, in the case of $m \approx l$ modes, near the equator. For a 100 μm sphere the effective mode volume is typically less than $1000 \mu\text{m}^3$ and for a 10 μm sphere it is less than $100 \mu\text{m}^3$ ⁵⁸. There are other important consequences of using very small microspheres. Figure 21a shows a WG mode spectrum for a microsphere with a radius $a = 6.8 \mu\text{m}$ as determined by the measured free spectral range $\Delta\lambda = 10.35 \text{ nm}$. The combination of the inverse dependence on radius and the increased ellipticity typically found in very small microspheres leads to very large splitting between different m modes. In this case the m mode splitting is $\Delta\lambda = 0.96 \text{ nm}$ implying $e = 1.5\%$. The fact that the maximum value for l , and thus m , is smaller for the same wavelength in small spheres means that $m \neq l$ modes are inclined more with respect to the equator than in larger spheres. In this 6.8 μm radius sphere $l_{\text{max}} = 77$ which means that the $l = 77, m = 76$ mode will be inclined by $\theta = \cos^{-1}(m/l) = 9.2^\circ$. This means that in addition to the large free spectral range and m mode splitting, fewer modes are excited with the same range $\Delta\Theta$ of input angles. The backscattering induced mode splitting can also be quite large in small microspheres as indicated by (1.73). The 6.8 μm radius sphere exhibits splitting 15 times greater than the resonance linewidth ($Q = 2 \times 10^7$) (fig. 21b). This can be important for cavity QED experiments if one is interested in coupling an atom to a single WG mode.

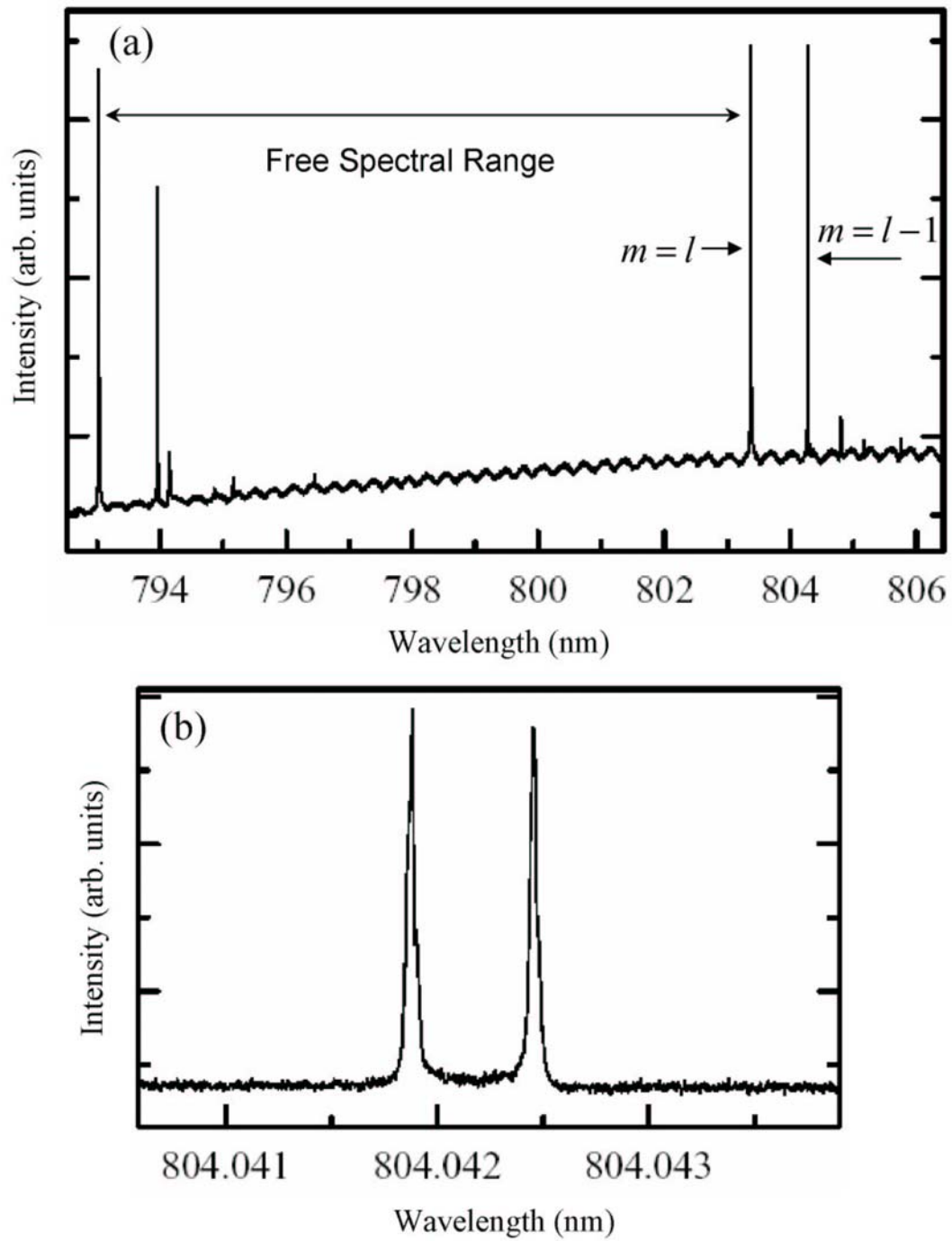


FIGURE 21. WG mode spectrum of 6.8 μm radius sphere. (a) The free spectral range of this small sphere is much longer than that shown in figure 17a. Large m mode splitting is also visible. (b) Backscattering causes stronger splitting of each m mode in a small sphere.

Unfortunately, small microspheres are also more sensitive to thermal effects compared to large spheres. Typically about 10% of the attenuation observed in fused-silica is due to absorption. This absorbed light is predominantly converted to heat. A small microsphere attached to the end of a long, thin, glass stem can not efficiently dissipate the heat produced by this absorption. The increase in temperature causes an expansion of the sphere and an increase in the index of refraction of the glass. Both of these effects tend to shift the resonance to longer wavelengths. This effect can be employed as a means of tuning WG modes, however, it also places a low limit on the amount of light that can be stored in the microsphere without significantly shifting a resonance.

Figure 22 shows a backscattering split, doublet mode that exhibits thermal bistability. When the mode is excited by scanning the pump laser with increasing wavelength (black) the resonance shift causes the mode to follow the laser wavelength leading to a significantly broadened and distorted lineshape. Once the laser wavelength slightly exceeds the thermal shift range of the mode, determined by the input power, the energy in the mode decreases allowing the sphere to cool and rapidly the resonance returns to its location at ambient temperature. This rapid shift causes a very rapid drop in output intensity to the background level on the long wavelength side of the resonance. When the pump laser is scanned with decreasing wavelength some subtle changes in line shape are evident. As the pump wavelength approaches the resonance from the long wavelength side there is initially a gradual increase in light coupled into the sphere. This is in contrast to the sharpness of the long wavelength side when the laser wavelength is

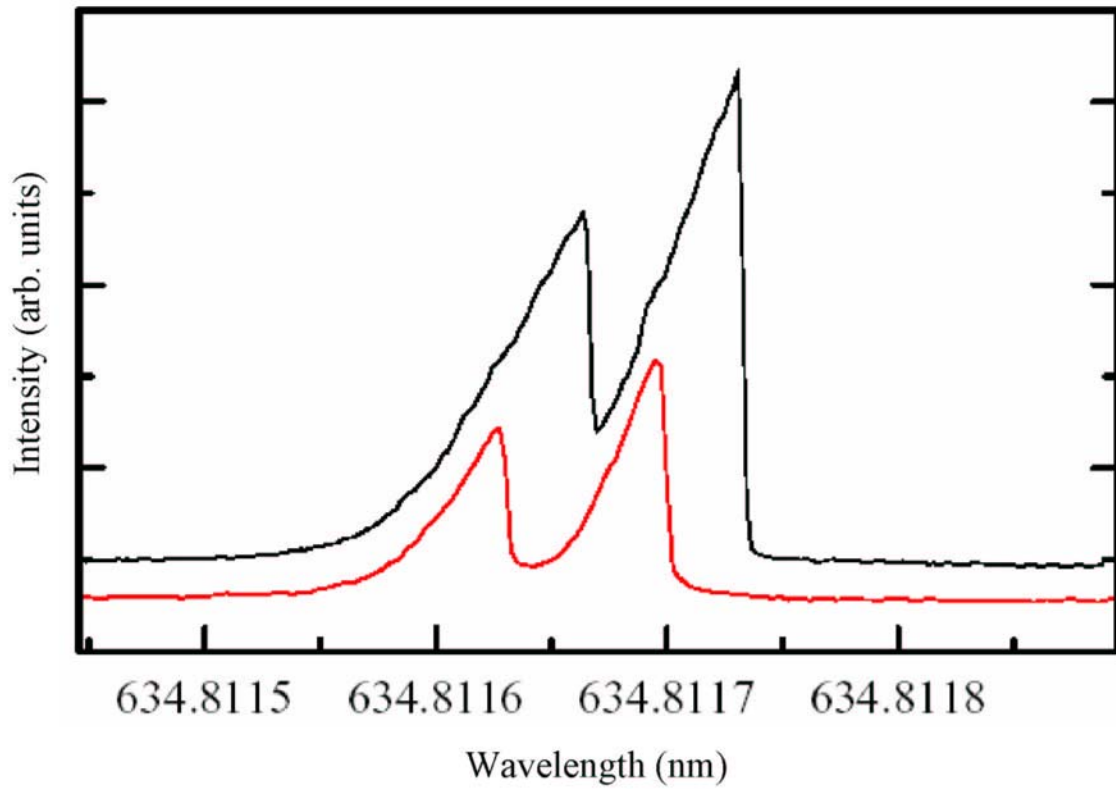


FIGURE 22. Thermal bistability. Heating, due to absorption of light from the excited mode, causes the resonance to follow the pump laser wavelength. The resonance lineshape is distorted slightly differently for increasing wavelength (black) and decreasing wavelength (red) scans. The black curve has been shifted vertically.

increasing. Then, as the intensity inside the sphere increases the heating causes the resonance to rapidly shift through the pump wavelength. There is a brief flash of intense light from the sphere, which is not resolved in figure 22, as the WG mode resonance

crosses the laser wavelength. As the laser wavelength continues to decrease the intensity in the WG mode gradually decreases allowing the sphere to cool and the resonance to return to its original location.

The threshold pump power for producing thermal bistability is sensitive to the sphere size and the Q-factor of the relevant mode⁵⁶. For a 30 μm radius microsphere with $Q=10^7$ threshold pump power has been measured at $P_{th} \approx 5 \mu\text{W}$. The threshold power can be much lower for modes with larger Q or smaller size. Furthermore, the threshold power can also be lower in vacuum where conduction through the stem becomes the only available heat sink.

Measuring Emission Directionality

Whispering gallery modes can be launched in the equatorial plane (y-z plane) of deformed microspheres using the same prism coupling technique as with standard microspheres. One would expect modes launched in this plane, which contains the approximate symmetry axis of the resonator, to precess. In practice we find that this does in fact occur in some of the deformed microspheres which we have made. When this precession occurs the light scatters strongly off the stem leading to only very low Q resonances being observed. The fact that some deformed microspheres do not exhibit this precession must be explained by the stem breaking the rotational symmetry strongly enough to stabilize modes near the equatorial plane. This effect can be reproduced in 3D

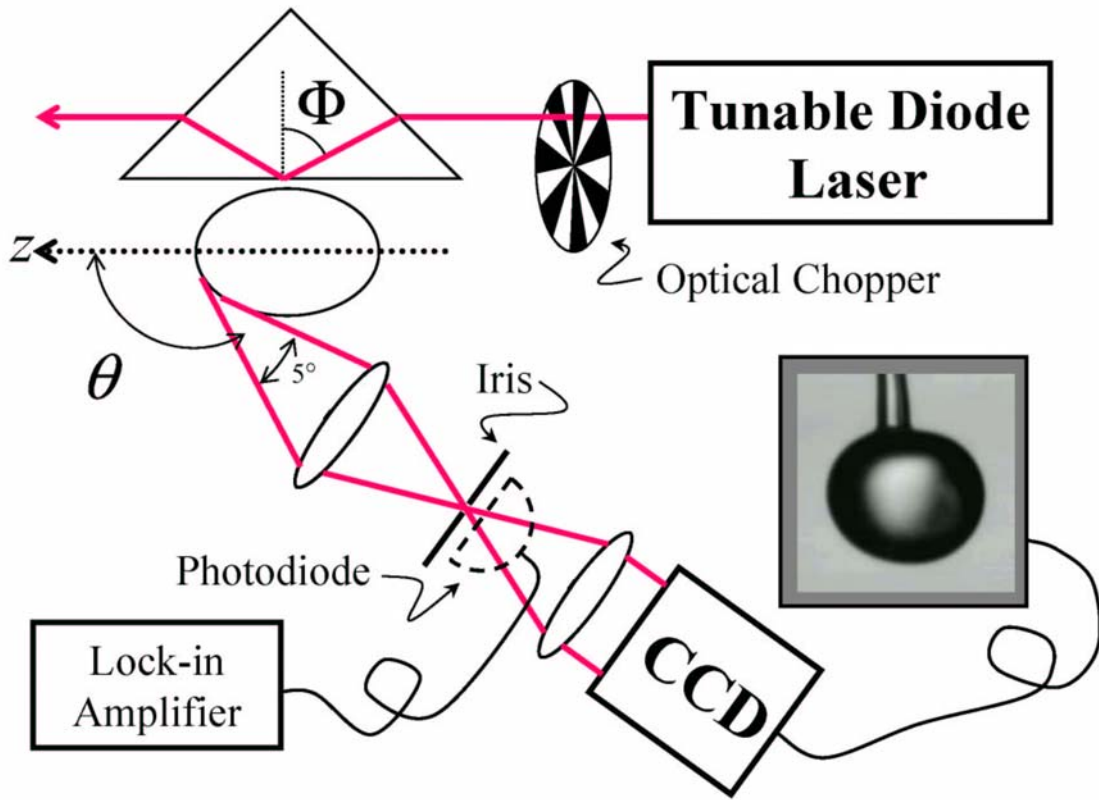


FIGURE 23. Experimental setup for measurement of emission directionality. A tunable diode laser excites WG modes by frustrated total internal reflection in a high index prism. Imaging optics are attached to a rail (not shown) which allows them to pivot around the microsphere. WG spectra are recorded by inserting a photodiode behind the iris, which is placed at the focal plane of the imaging objective. (The size of the microsphere is greatly exaggerated to indicate orientation.)

ray simulations by including a deformation in the x-y plane of the resonator. All experiments presented in this dissertation were performed with deformed microspheres which did not demonstrate strong scattering at the stem and for which the emission pattern was confined near the equatorial plane.

Emission directionality is of primary interest with deformed microspheres both for practical applicability and because it is an indicator of the internal dynamics of the

resonator. The far field emission directionality of deformed microspheres was measured by mounting imaging and detection optics on a rail which could be pivoted around a point directly below the microsphere. The measurement apparatus is depicted in figure 23. A microscope objective is positioned between one and two focal lengths away from the microsphere in order to produce a magnified real image. This magnified image is then imaged onto a CCD camera with a setup similar to that described for the microsphere fabrication monitoring system. A small iris is placed at the real image plane of the first objective lens and is used as a spatial mask allowing one to collect light coming only from a specific location on the microsphere. In order to collect resonance spectra a photodiode is mounted so that it can be rotated into position directly behind the iris. With this setup the rail can be positioned to collect in any direction within the 180° range not blocked by the prism, and the iris can be adjusted to collect only light emitted tangentially (not surface scattered light) from the sphere. By taking resonance spectra from different directions, θ (measured from the z axis of the microsphere), one can piece together the far field emission pattern of the deformed microsphere. The $f/\#$ of the objective lens determines the maximum angular resolution that can be obtained, in our case typically 5° .

CHAPTER V

RAY DYNAMICS IN DEFORMED FUSED-SILICA MICROSPHERES

In chapter III we introduced the formalism of Hamiltonian dynamics applied to near-integrable systems. The technique of visualizing system dynamics using Poincaré surface of section plots was presented and the predictions based on ray tracing models of asymmetric dielectric resonators were discussed. We showed that two dimensional billiard models make robust predictions regarding the optical properties of ARCs, which are largely insensitive to the exact shape of the resonator. In this chapter we will present measurements of the optical properties of WG modes in deformed fused-silica microspheres. We will focus on determining the usefulness of 2D ray tracing models in understanding the experimental results. The resonators investigated have average radii that are hundreds of times larger than the wavelength of the input light and are therefore expected to be well described by geometrical optics. The limits of the 2D model will be discussed in light of specific discrepancies between the model predictions and experimental results. A fully three dimensional billiard model is developed and applied to dielectric resonators for the first time. This model demonstrates the qualitatively different dynamics present in three dimensional systems. The chapter concludes with a discussion of the possible role that Arnold diffusion plays in determining the lifetime of WG resonances in deformed microspheres.

Directional Emission from Deformed Fused-Silica Microspheres

A typical far-field emission pattern for a deformed microsphere is presented in figure 24. The average diameter of this microsphere was about 200 μm and it had a deformation $\epsilon = 0.037$ where ϵ is defined by equation (1.64) and corresponds to the deformation parameter of the quadrupole. WG modes were launched using a prism as described in Chapter IV such that $\sin \chi_0 \approx 1$. The emission pattern exhibits a strong peak at $\theta = 45^\circ$ in the far field. This emission was observed to emanate from the edge of the sphere when imaged, indicating that the light escapes nearly tangential to the surface at the position $\theta = -45^\circ$ on the resonator. Figure 25 diagrams the emission location and direction as viewed from above. Light escaping from the opposite point of the sphere was visible in the experiment due to reflection off of the prism face as indicated in figure 25. This light was blocked by the iris at the real image plane (see fig. 23) and is therefore not included in the data of figure 24. A less intense peak in the far field emission can be seen at $\theta = 135^\circ$. This light was also observed to emanate from the sphere tangentially as indicated by the dotted arrow in figure 24.

All deformed microspheres that we have observed demonstrate very similar far-field emission patterns. There are always two emission peaks in approximately the same directions as those shown in figures 24 and 25. The only significant variation is the relative intensity of the two emission peaks. In general the peak at $\theta = 45^\circ$ is more intense for strongly deformed microspheres and the two peaks tend to balance out at

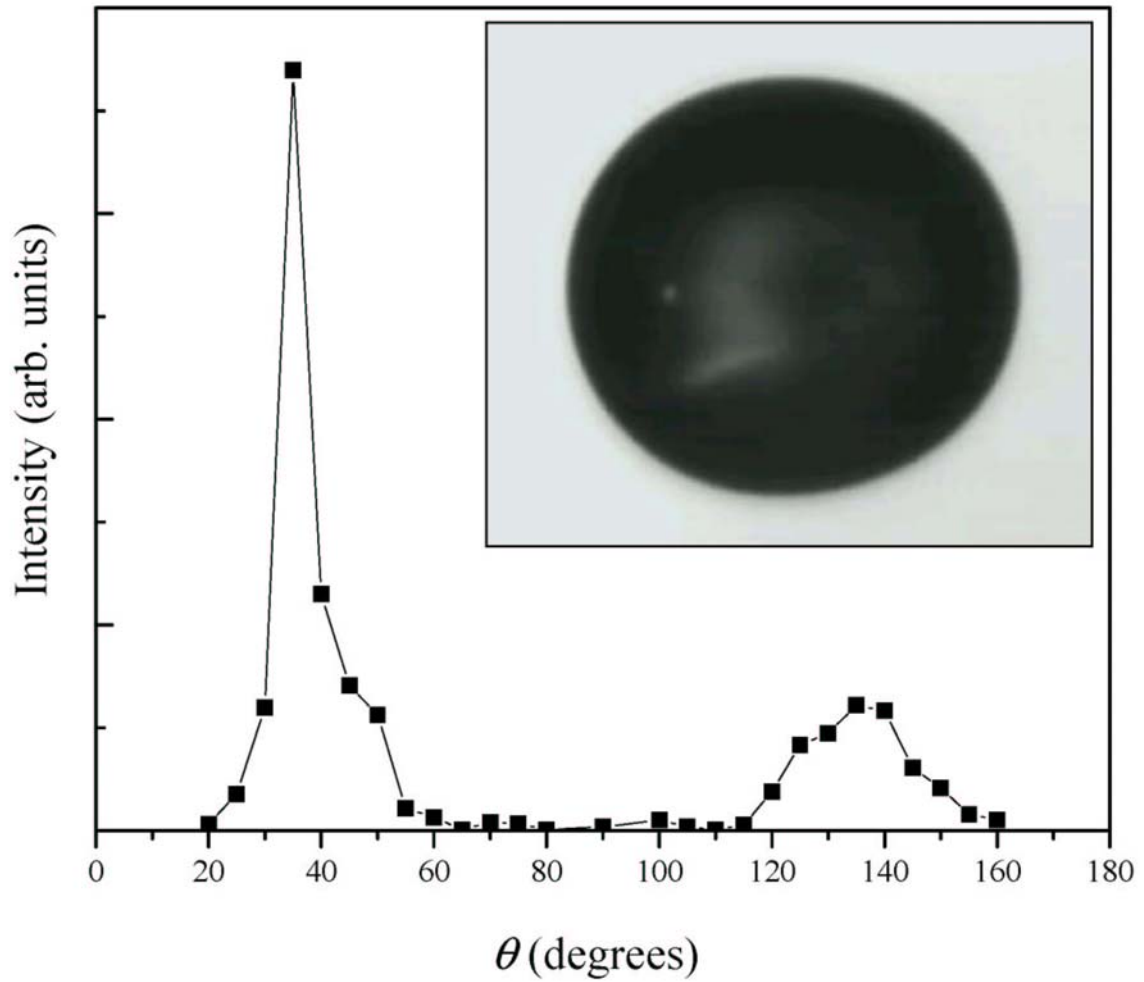


FIGURE 24. Directional emission from a deformed microsphere. The peak intensity is offset from 90 due to dynamical eclipsing. The eccentricity of the microsphere is measured from the image (inset) to be $\epsilon = 0.037$.

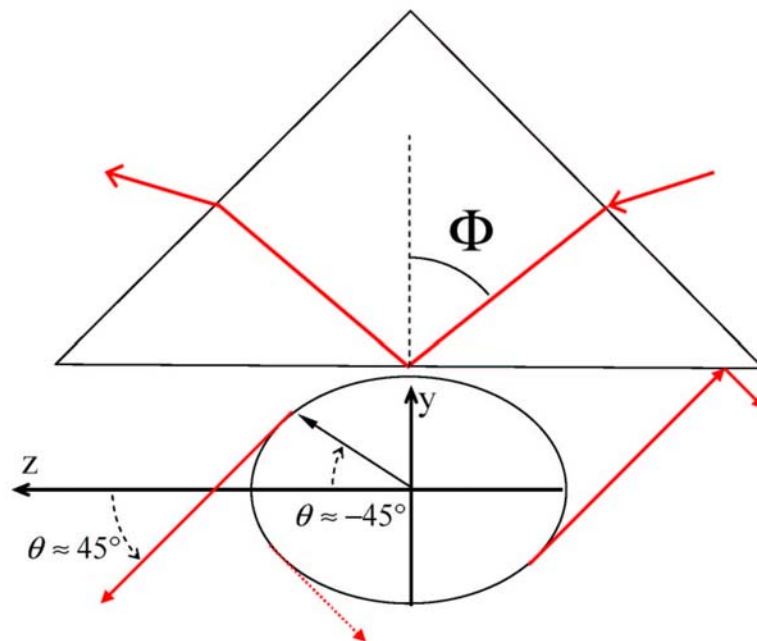


FIGURE 25. Schematic of observed emission directions from a strongly deformed microsphere. The light reflected off of the prism is not collected when measuring the far-field emission pattern.

smaller deformations. In this chapter we will show that the strong emission peak at $\theta = 45^\circ$ is well explained by the dynamical eclipsing effect in the ray model. The second strong emission direction cannot be explained by the ray model, however, and will be discussed in detail in Chapter VI.

The microsphere shown in figure 24 was repeatedly reheated in order to reduce its deformation. The far-field emission patterns produced by this microsphere at two smaller deformations are shown in figure 26. The emission pattern measured with a deformation of $\epsilon = 0.006$ shows two nearly equal intensity peaks at $\theta = 45^\circ$ and 135° . When the deformation was reduced to $\epsilon = 0.004$, however, the emission became nearly isotropic.

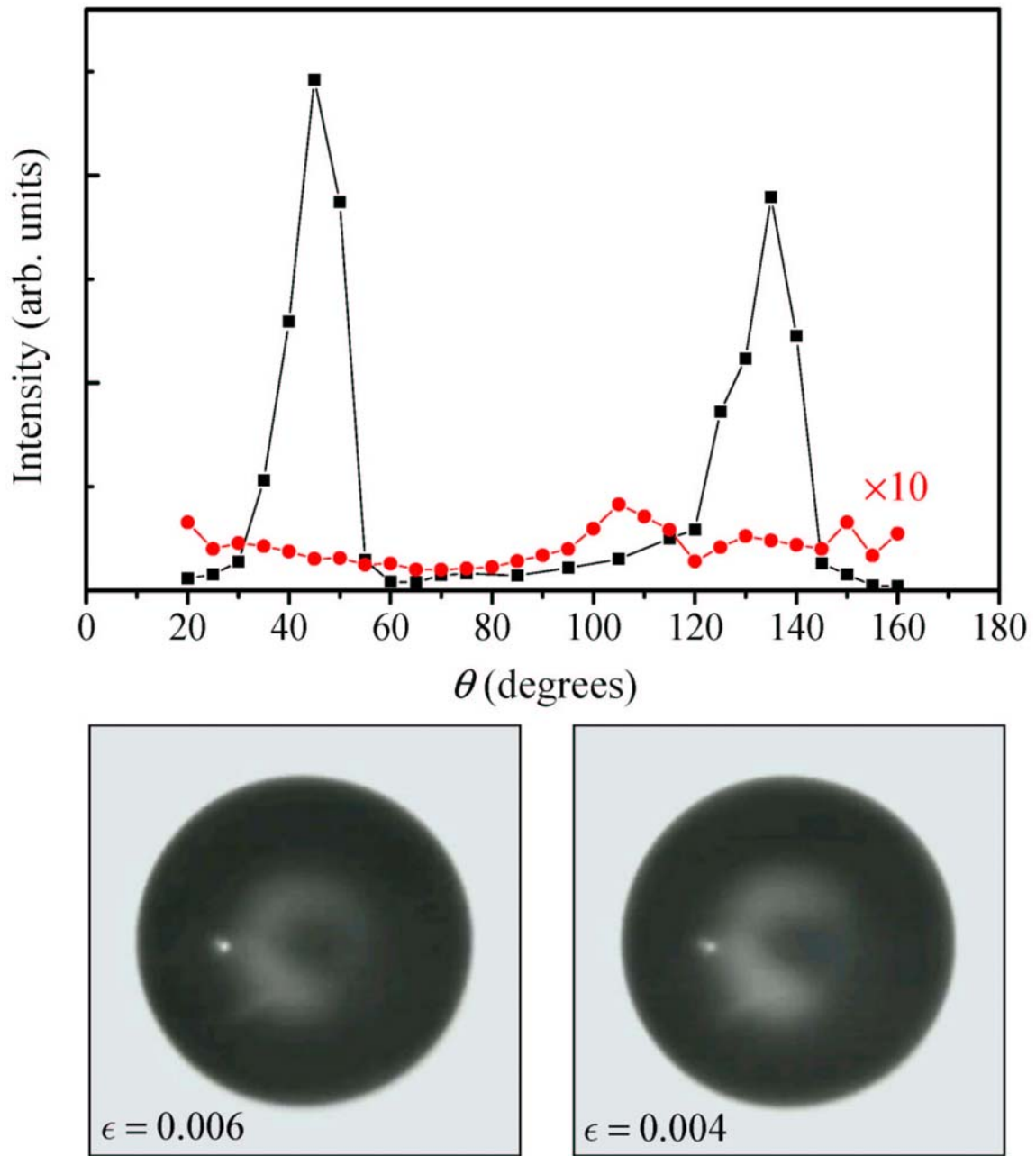


FIGURE 26. Far-field emission patterns for a deformed microsphere at two different deformations. At an eccentricity $\epsilon = 0.006$ (black squares) the microsphere exhibits nearly equally strong emission in two directions. When the eccentricity is reduced to $\epsilon = 0.004$ (red circles) the emission is nearly isotropic. The red curve has been magnified by a factor of ten.

The recovery of isotropic emission as the deformation was reduced to zero is an important verification that the directional emission was not somehow caused by the fabrication technique where two spheres were fused together.

Whispering Gallery Mode Lifetimes in Deformed Microspheres

In addition to recording the change in far-field emission pattern, we also measured the Q-factor of the WG modes at each deformation. In all cases we found that as the deformation was reduced the Q-factor of the WG modes increased. Figure 27 shows a plot of Q versus ϵ for the microsphere studied above as its deformation was reduced. The most dramatic increase in the Q-factor was the jump by nearly a factor of twenty that occurred when the emission became isotropic. The Q-factors measured for WG modes in different microspheres with the same eccentricity do not demonstrate the same universality as the far-field emission pattern. The highest Q measured for this microsphere was only 4×10^7 which is about an order of magnitude less than what we routinely measure for non-deformed microspheres. In fact, in Chapter VI we will present measurements of Q-factors as high as 7×10^7 for WG modes that display directional emission. We attribute the low ultimate Q for this microsphere to the fact that it was reheated nearly twenty times over the course of a week. Repeated accumulation of contaminants on the sphere degraded the quality of the glass to the point that there was noticeably more light scattered from the surface of the sphere. It should be noted however that the measurements were performed in the order of

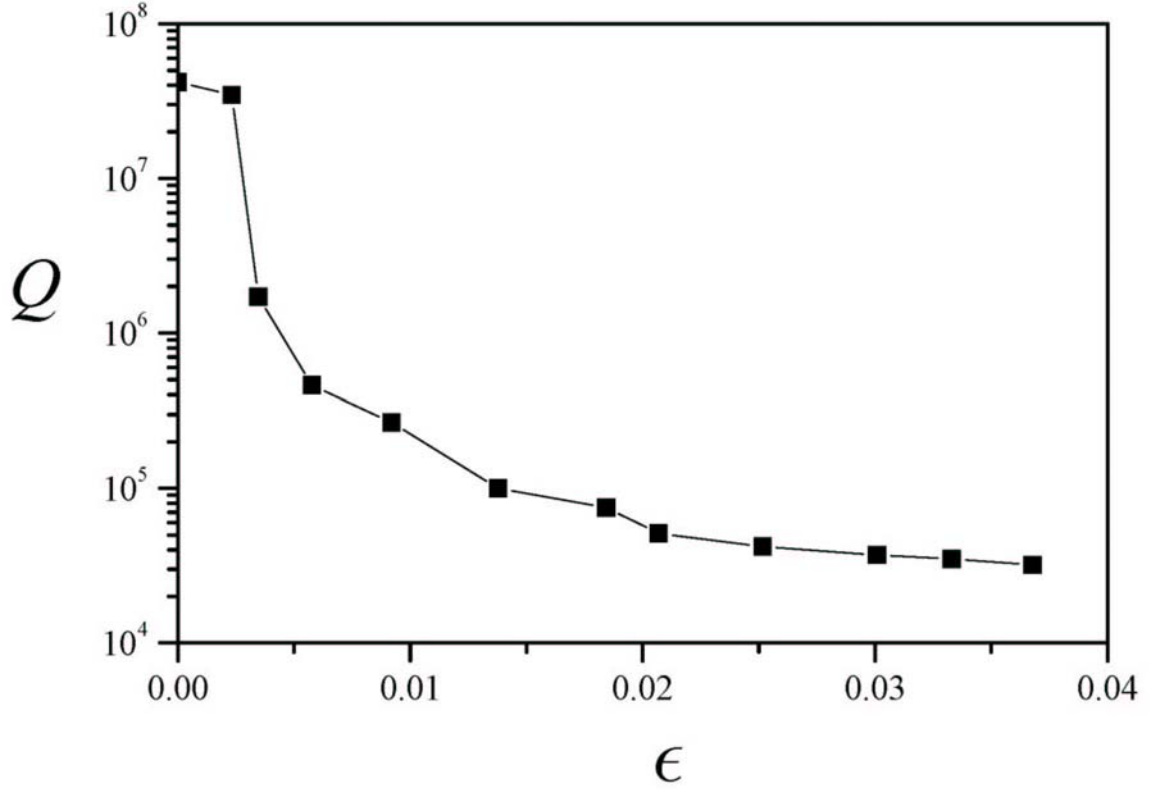


FIGURE 27. Q-factor versus deformation for a single deformed microsphere.

decreasing deformation, and thus the significant increases in Q-factor were observed despite the fact that the optical quality was being slowly degraded.

In order to further probe the dynamics WG modes in asymmetric resonators we also launched light into the microsphere with different initial angles of incidence. This was accomplished by changing the angle of incidence Φ of the focused laser beam at the surface of the prism as described in the previous chapter. For a deformed microsphere with $\epsilon = 0.02$ directional emission was observed from WG modes with $Q = 3 \times 10^7$ when the excitation was such that $\sin \chi_0 \geq 0.99$. To ensure that only these modes were excited

the entire cone of the focused pump laser was incident on the prism face at $\Phi > \sin^{-1}(n_s/n_p)$. This configuration sacrifices input coupling efficiency but is strongly selective to exciting only maximum l modes. When the input beam was adjusted to provide smaller $\sin \chi_0$, the same emission directionality was always observed, however, some of the modes were observed to have reduced Q-factors. Figure 28 shows a plot of the lowest Q mode observed as a function of $\sin \chi_0$. Higher Q modes were observed

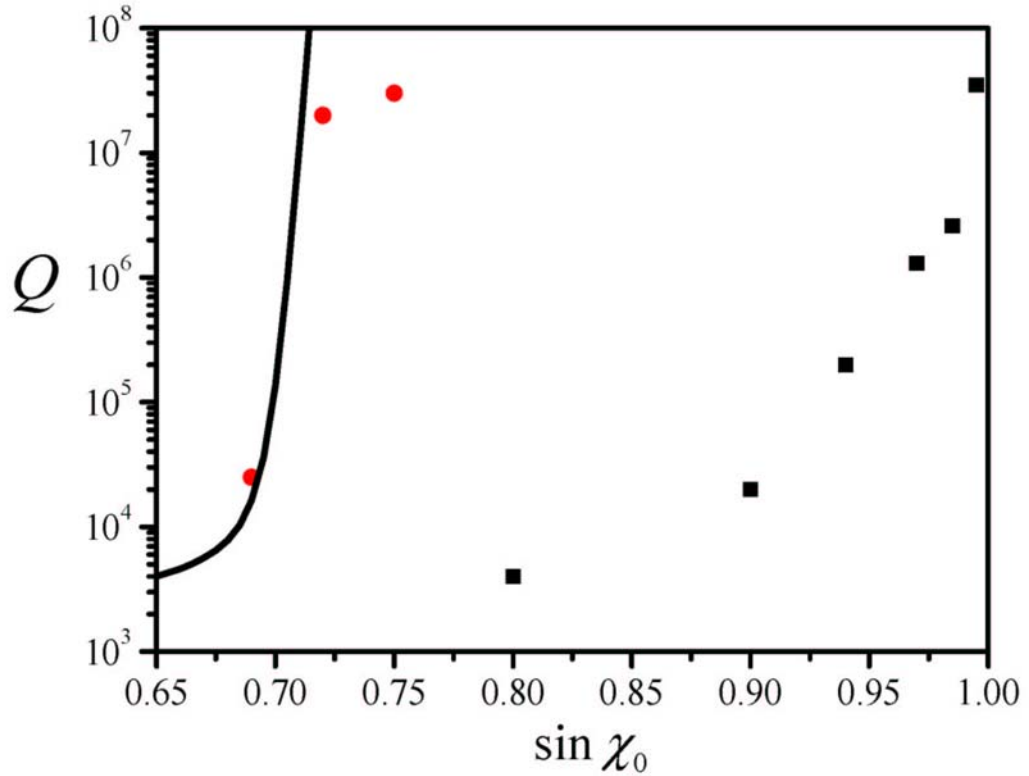


FIGURE 28. Q-factor versus $\sin \chi$ measured in a $\epsilon = 0.01$ deformed microsphere (black squares). Measurements (red circles) and a theoretical calculation (black curve) of Q-factors in a 200 μm non-deformed microsphere are shown for comparison.

simultaneously and will be discussed in more detail in Chapter VII. The values reported for $\sin \chi_0$ represent an estimate of the lowest value launched by the focused input beam. The lowest observed Q-factor decreased by over three orders of magnitude as the input condition varied over the range $1 \geq \sin \chi_0 \geq 0.9$. The far-field emission pattern was unaffected by the change of initial condition indicating that the light escaped from the same region of phase space in all cases. In light of this, the roughly exponential decrease in mode lifetime indicates that the rate of transport through phase space is strongly sensitive to $\sin \chi$. Furthermore it shows that rays launched at $\sin \chi \approx 1$ spend by far the largest portion of their lifetime with $\sin \chi > 0.9$.

It should be noted that in a 200 μm diameter, non-deformed microsphere, WG modes were observed to have $Q > 10^7$ down to $\sin \chi_0 = 0.73$. Three Q-factor measurements in a 200 μm sphere are included in figure 28 for comparison to the deformed microsphere. In addition, the curve indicates the predicted Q-factor as a function of $\sin \chi$ for WG modes in a 200 μm sphere. The theoretical prediction is derived from an expansion of equation (1.9) for large size parameters. Because it is derived from the solution of Maxwell's equations it assumes no material losses and thus predicts impossibly high Q-factors for $\sin \chi > 0.72$.

A possible explanation for the observation of directional emission and Q-spoiling in deformed microspheres is the process by which they fabricated. One might worry that the seam where the two spheres were fused together could cause both effects. This explanation is partially dismissed by the observation that high Q-factors and isotropic

emission are recovered as the deformation approaches zero. This could, however, be seen as evidence that the effect of the seam is merely being reduced by the continued heating of the microsphere. The conclusive evidence that joining the two spheres does not cause Q-spoiling and directional emission actually comes from a slightly different type of deformed microsphere. We can fabricate deformed microspheres that are elongated along the direction of the stem. In this case the two spheres are placed in contact such that the centers of the two spheres are in line with the stem. Deformed microspheres fabricated in this way maintain rotational symmetry about the stem axis. Due to the presence of the stem, it is not possible to excite WG modes in the plane with the largest cross-sectional deformation. Instead we launch WG modes at an inclination of approximately 45° to this plane. Because the microsphere is axisymmetric these modes precess around the long axis. Even for large eccentricities, these inclined modes in axisymmetric deformed microspheres do not demonstrate directional emission. Furthermore, the Q-factors of the inclined modes are typically near 10^8 and, as shown in figure 29, are nearly identical to those excited in the plane perpendicular to the stem with circular cross section. These observations demonstrate that fusing two microspheres does not produce a defect which causes Q-spoiling and directional emission.

Instead, these measurements indicate that the destruction of rotational symmetry by the presence of the stem may be the cause of these effects. As discussed in Chapter III, the dynamics of an axisymmetric resonator can be reduced to those of a corresponding 2D billiard. This cannot be done for a nonaxisymmetric resonator, however. Later in this chapter we will show that directional emission and Q-spoiling are

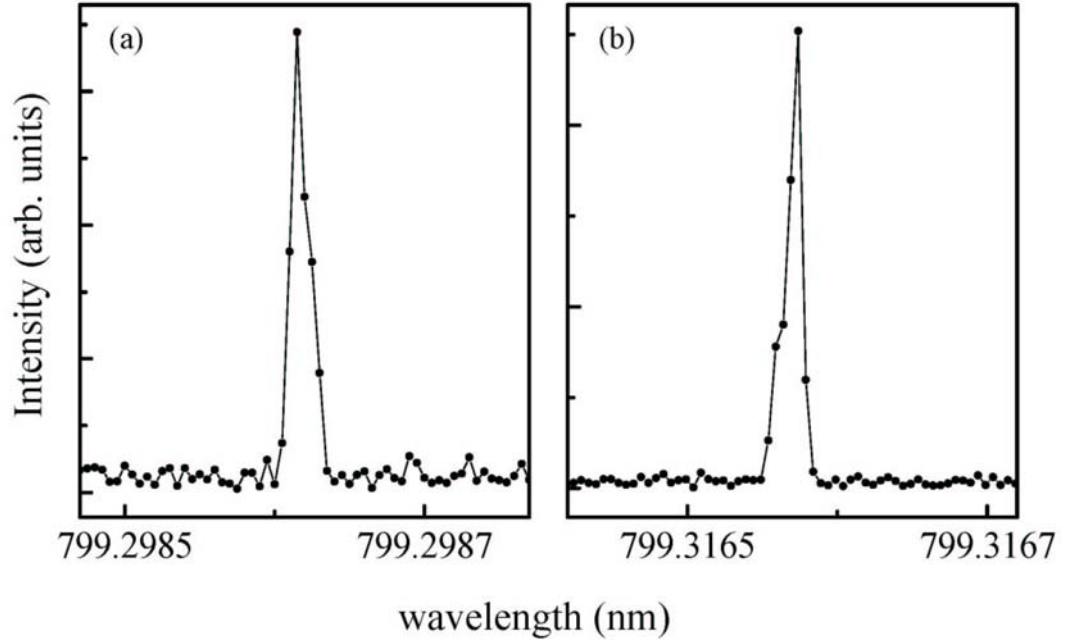


FIGURE 29. High-Q WG modes in axisymmetric deformed microsphere. WG modes were launched in planes inclined 45° from the stem (a) and perpendicular to the stem (b). In both cases the Q-factor was measured to be 5×10^7 .

not expected for a 2D resonator in the range of eccentricities that we have studied experimentally. In a 3D nonaxisymmetric resonator, on the other hand, qualitatively different dynamical processes occur, which can produce the experimentally observed phenomena.

Emission Directionality as a Probe of Ray Dynamics

For oval shaped billiards, refractive escape is always expected to occur at two narrow regions on opposite sides of the resonator. Normally these escape regions are located at the points of highest curvature as intuitively expected. It is possible, however,

for a stabilized periodic orbit located at these points to displace the escape regions away from the points of highest curvature. This dynamical eclipsing effect was found to always shift the escape regions such that rays escape just before passing the highest curvature points. Dynamical eclipsing can be considered to be an exception to the universal emission directionality predicted for all oval cavities. Whether or not dynamical eclipsing occurs in a particular resonator, however, the directionality of refractive emission is largely insensitive to the initial conditions of rays in the cavity. The conditions of refractive escape are entirely determined by the local dynamics in the $\sin \chi \approx \sin \chi_c$ region of phase space. In this way, the emission directionality of an asymmetric resonator is a probe of the ray dynamics in this region of phase space.

As demonstrated in figure 7, the ellipse is the only oval billiard that is completely integrable and therefore has no chaos. As such, refractive escape from an elliptical cavity will only occur if the ray starts on an invariant curve which crosses the critical line $\sin \chi = \sin \chi_c$. Rays launched in an ellipse will either never escape refractively, or escape rather quickly by refraction. The direction of the escape would, as in any case where a non-chaotic trajectory is launched, depend on the specific initial condition. The insensitivity to initial conditions occurs when chaotic diffusion is required for the ray to get from its starting condition to the critical line. Because chaotic diffusion is typically slow relative to orbiting motion, chaotic trajectories in an oval resonator can reasonably be considered to follow invariant curves of the ellipse for short times and gradually move from one IC to another with a trend towards smaller $\sin \chi$. Fast motion along the ICs

guarantees that refractive escape will occur at the minima of the IC that is tangent to the critical line. Therefore, the generic case of refractive emission from the points of highest curvature in a non-integrable resonator is nearly identical to the emission produced from an elliptical resonator when rays are launched on the tangent IC.

When refractive escape is observed away from the expected locations, this is a clear indicator of the presence of stable islands that are preventing chaotic trajectories from reaching the minima of the tangent IC. For an oval resonator we know that these islands must be located at $\theta = 0$, and $\pi^\dagger$. The index of refraction of the resonator and the surrounding material determine the critical angle for total internal reflection. If these are known, then the locations in phase space of the eclipsing stable islands are known. This information can be used to determine some details regarding the type of deformation the resonator might have.

In the case of a fused-silica ($n = 1.45$) resonator in air ($n \simeq 1$) the critical angle is $\sin \chi_c = (1/1.45) = 0.69$. This is a fortunate value for using emission directionality to determine the type of deformation because it is very near the unperturbed location of the strong four bounce islands ($\sin \chi = \sqrt{2}/2 \approx 0.71$) that are present in most oval billiards. All oval deformations of the ellipse have stable four bounce islands but not always in the same locations. In order to deform an ellipse while maintaining a constant aspect ratio one must choose whether the curves connecting the ends of the major and minor axes

[†] In the spherical polar coordinates we are using for the 3D case θ corresponds to ϕ from the 2D case. To avoid confusion, for the remainder of this dissertation we will use θ for both cases.

would be pushed out or in relative to the ellipse. Ovals for which the curves are pushed out, such as the quadrupole, tend to stabilize the diamond shaped four bounce orbit while ovals which push the curves in tend to stabilize the rectangular four bounce orbit. As previously demonstrated, the quadrupole deformation

$$r(\theta) = 1 + \epsilon \cos(2\theta) \quad (5.1)$$

stabilizes the diamond orbit. Adding a sextupole term,

$$r(\theta) = (1 - p) + \epsilon \cos(2\theta) + p \cos(4\theta) \quad (5.2)$$

with $p > 3\epsilon^2/4$ can stabilize the rectangular orbit instead. The condition on p arises from the fact that equation (5.2) is a good approximation to an ellipse for $p = p_e \approx 3\epsilon^2/4$.

Figure 30 shows one quadrant of an ellipse, a quadrupole, and a quadrupole/sextupole, each with an aspect ratio $(1 - \epsilon)/(1 + \epsilon)$ where $\epsilon = 0.06$ and $p = 2p_e$. This choice for p produces an oval that is deformed from the ellipse approximately the same amount as the quadrupole, but in the opposite direction. Figure 31 compares SOS plots for the two non-elliptical ovals demonstrating the difference in the stabilization of the period four orbits. Although the two resonators appear very similar and have the same aspect ratio, only the quadrupole would be expected to demonstrate dynamical eclipsing, because in the quadrupole/sextupole there is not a stable island at the minima of the tangent IC.

The emission pattern shown in figure 24 clearly demonstrates dynamical eclipsing. This fact indicates that there must be a stable island located near $\sin \chi = \sin \chi_c$, $\theta = 0, \pi$ suggesting that the microsphere has a quadrupole type

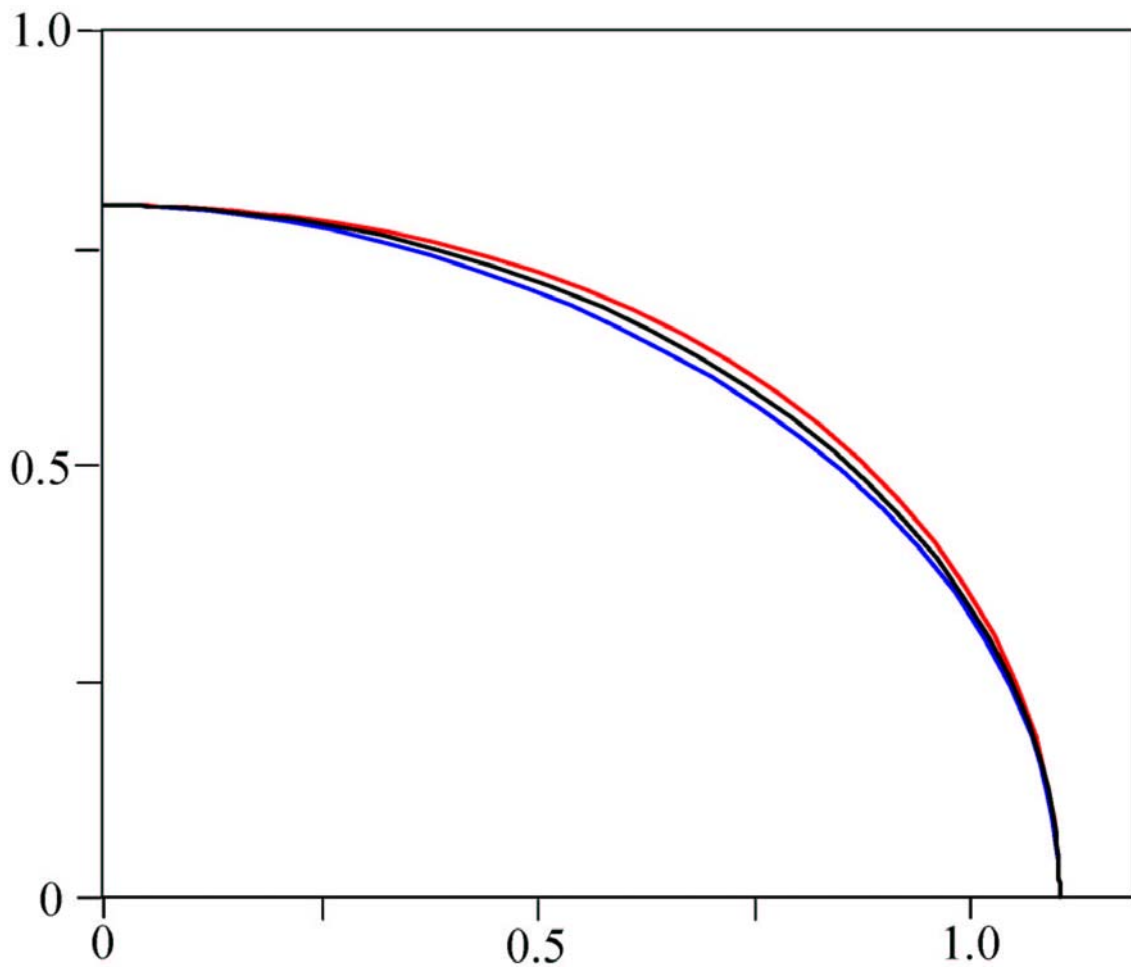


FIGURE 30. Two deformations of the ellipse. The quadrupole (red) lies outside the ellipse at all points, whereas, the quadrupole/sextupole (blue) lies inside the ellipse at all points.

deformation. This assessment is also supported by the fact that the minimum radius of curvature is larger for the quadrupole shape than the other two shapes considered. This makes the quadrupole a more energetically favorable shape as the original bi-sphere shape relaxes towards a single sphere. This is particularly true for large aspect ratios, shortly after the resonator becomes completely convex.

Whispering Gallery Mode Lifetimes in the 2D Ray Model

The WG modes, for which the emission directionality was plotted in figure 24, were measured to have a Q-factor of about 3×10^4 . In a $200 \mu\text{m}$ diameter resonator this means that escape occurred after only about six round trips. The light was launched with $\sin \chi \approx 1$ and escaped at $\sin \chi \approx 0.69$. This requires a very rapid diffusion through phase space, and seems to imply strongly chaotic dynamics that are largely unimpeded by islands and KAM curves. This cannot be the case, however, since Lazutkin's theorem guarantees that unbroken KAM curves remain at large values of $\sin \chi$ for any completely convex billiard. These unbroken KAM curves are evident in the SOS plot for both of the shapes shown in figure 31. One can consider shapes which might exhibit more chaos than the quadrupole for the same aspect ratio. For instance, since setting $p = p_e$ in (5.2) deforms the quadrupole back towards the integrable ellipse, setting $p = -p_e$ will create a billiard with twice the deformation relative to the ellipse than the quadrupole. Figure 32 shows SOS plots for $\epsilon = 0.06$ and $p = p_e$, 0 , and $-p_e$. Whereas the resonator with $p = p_e$ has very little chaos and closely resembles the SOS of the ellipse, the resonator with $p = -p_e$ exhibits significantly more chaos than the quadrupole. Despite this, however, KAM curves still remain near $\sin \chi \approx 1$. Even this strongly chaotic billiard predicts an infinite lifetime for WG modes launched with the conditions of the experiment.

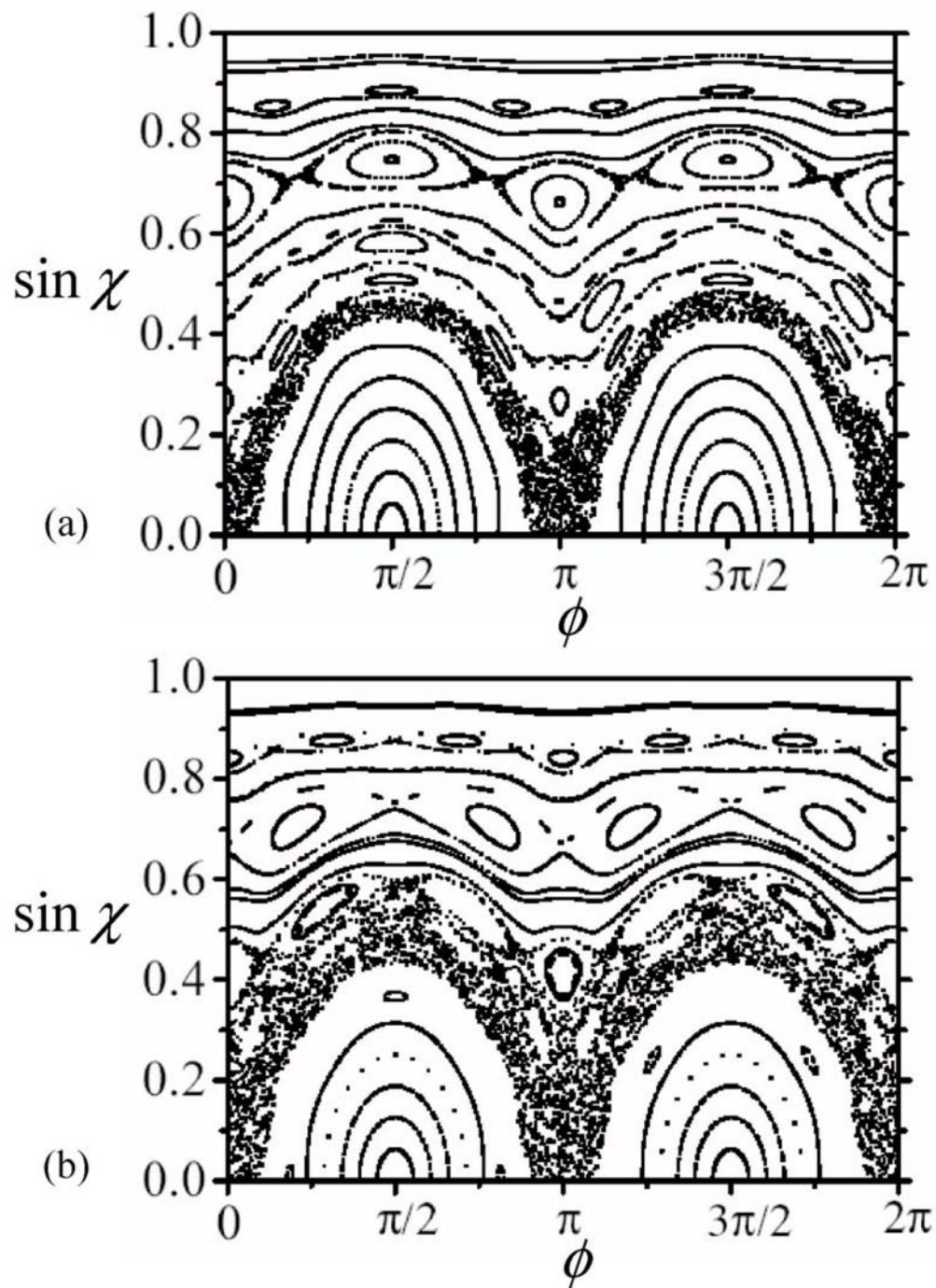


FIGURE 31. Poincaré surfaces of section for quadrupole (a) and quadrupole/sextupole (b) of the same aspect ratio and approximately same perturbation from the ellipse. The dominant difference between the two shapes is which set of period four islands are stabilized.

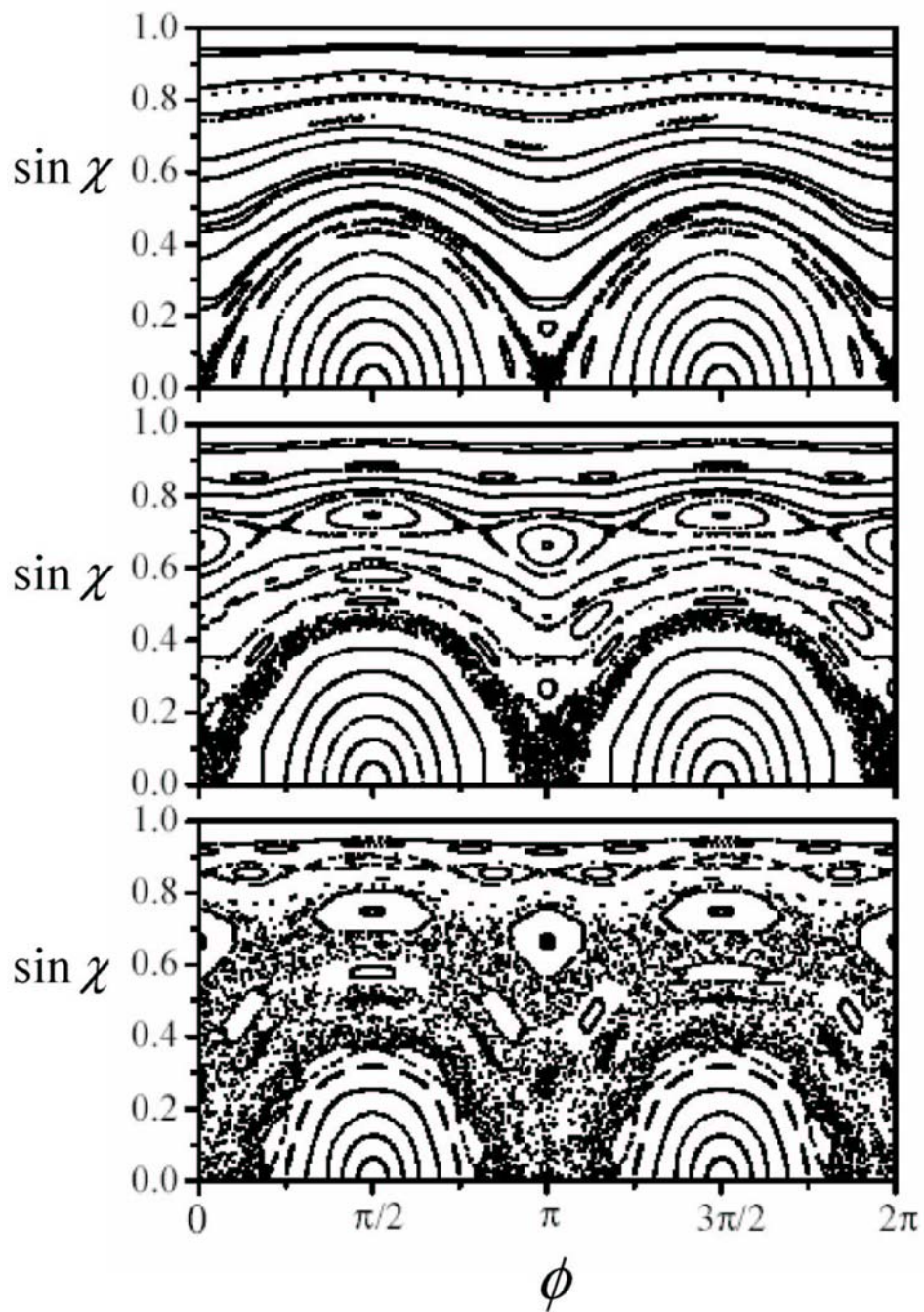


FIGURE 32. Poincaré surfaces of section for quadrupole (center), and two deformations of the quadrupole. The top deformation of the quadrupole makes the shape nearly elliptical as evidenced by the lack of chaos. The bottom surface of section corresponds to the same magnitude perturbation to the quadrupole but in the opposite direction.

The issue of phase space transport is even more problematic for microspheres with smaller deformations. For each of these deformations the microsphere exhibited a directional emission pattern. The directional emission pattern recorded for $\epsilon = 0.006$ in figure 26 indicates that the escape occurred near $\sin \chi \approx \sin \chi_c$. This demonstrates that significant phase space transport occurs even at small deformations where the phase space region between the initial and escape conditions for a ray is densely filled with KAM curves. For weakly deformed microspheres ($\epsilon \approx 0.01$) there is no reasonable smooth shape that produces nearly enough chaos in a 2D billiard to explain the observed emission properties.

One mechanism that might be suggested for how light launched at $\sin \chi_0 \approx 1$ can reduce its angular momentum to the point of $\sin \chi \approx \sin \chi_c$ is surface roughness. It is well known that extrinsic noise can cause diffusion in a Hamiltonian system which would otherwise not allow diffusion. In the case of an asymmetric resonator, surface roughness would introduce a random perturbation to a trajectory. This would allow a trajectory to move from one KAM curve to another, for instance. A very similar effect is of concern when computer ray tracing models are run for many iterations. In this case the random perturbation is due to rounding errors. The similarity between these two processes suggests that the effect of surface roughness on escape time can be simulated by increasing the rounding error in a ray tracing model. Figure 33 shows a single trajectory launched at $\sin \chi_0 = 0.95$. To generate this figure, the specified accuracy of the root finding algorithm that determines the location of the next reflection was reduced to 10^{-5} from 10^{-15} as it is normally run. In this simulation the ray escaped the resonator after

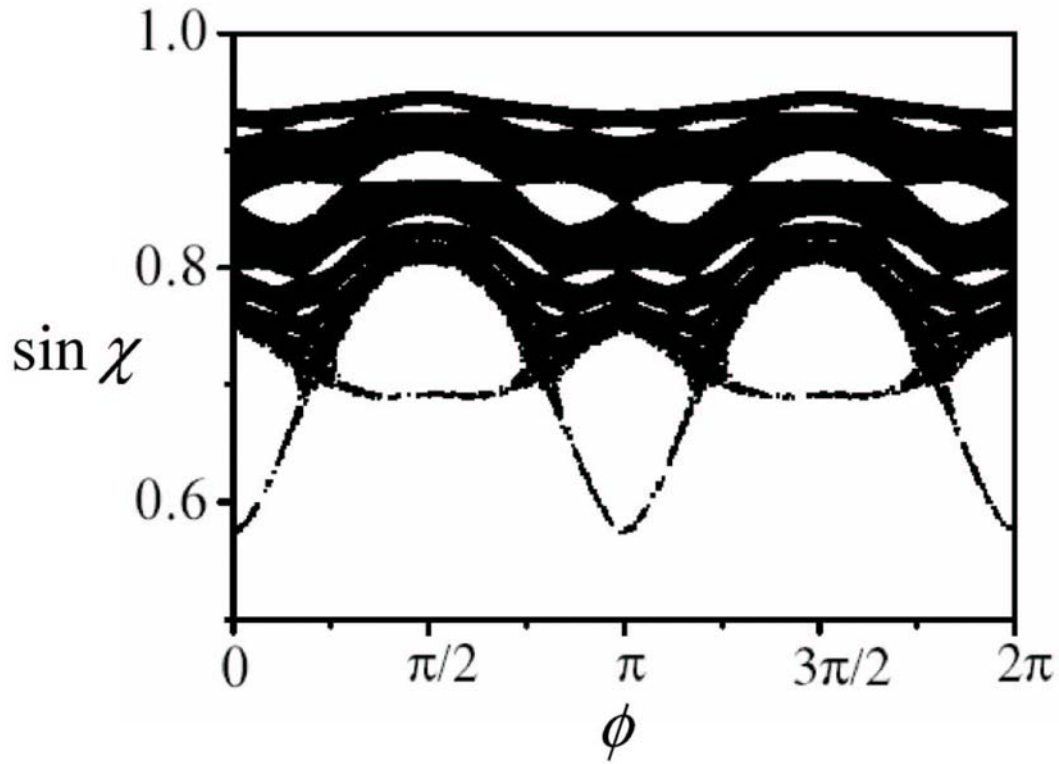


FIGURE 33. Poincaré surface of section for a “rough” quadrupole. A single trajectory if followed for 10^5 reflections with the accuracy of the calculation intentionally reduced to simulate surface roughness. The trajectory is observed to steadily move towards smaller $\sin \chi$ at a roughly constant rate.

90,000 reflections. It is important to notice however that the diffusion rate remained very nearly constant throughout the 90,000 reflections. This does not agree with the observation that the diffusion rate in deformed microspheres is strongly dependent on $\sin \chi$. Furthermore, the amount of surface roughness is not expected to be greater in a deformed microsphere than a regular microsphere. If surface roughness were a significant factor in microspheres, WG modes with $Q > 10^7$ would not be observed at

$\sin \chi_0 = 0.73$. Furthermore, we have found that high-Q modes persist in strongly deformed microspheres formed with the long axis in line with the stem. This confirms that the Q-spoiling is not due to defects caused by fusing two spheres together. Instead, it suggests that the unknown mechanism of phase space transport in deformed microspheres is related to symmetry.

It is clear that the ray tracing model cannot explain the observed phase space transport in weakly deformed resonators. The asymmetric boundary alone is not sufficient to allow rays to change angle of incidence to the degree necessary to explain the observed emission patterns. For WG mode excitation near $\sin \chi \approx 1$ the billiard model will always predict isotropic emission and high Q-factor up to a threshold deformation at which point emission becomes highly directional and Q decreases rapidly with increasing deformation. There are, however, two major differences between the experimental system and the theoretical model considered thus far. First, despite the large size parameter of the deformed microsphere, wave effects cannot be completely neglected. This is emphasized by the fact that the directional emission data is derived from measurements of WG resonance spectra. Second, the light coupled into a deformed microsphere will follow a trajectory in three dimensions, which cannot necessarily be reduced to an equivalent two dimensional billiard. Consideration of the impact of wave effects on the optical properties of deformed microspheres is deferred to Chapter VI. In the remainder of this chapter we will explore the effect of adding a third degree of freedom within the ray model.

Three Dimensional Ray Tracing Model

Arnold diffusion, whereby rays can circumvent KAM curves in a 3D resonator, is a possible mechanism for light to achieve the required change in χ . In order to test whether Arnold diffusion plays a role in the diffusion of trajectories in non-axisymmetric 3D ARCs we developed a 3D ray tracing model. Our model is conceptually a simple extension of the previously discussed 2D ray model to three dimensions. In practice this extension introduces a number of new challenges. In addition to the increased numerical difficulty associated with solving a system of nonlinear equations (which can be greatly simplified in a 2D resonator), one is confronted with the practical difficulties of accurately describing the 3D resonator and creating a useful representation of the 6D phase space.

It is prohibitively difficult to determine the exact shape of a spheroid. With a 2D resonator one is able to image the object and fit a curve to the perimeter. In order to completely determine the shape of a 3D resonator one would require an infinite number of 2D images. The three orthogonal views of the deformed microsphere shown in figure 15 determine three orthogonal cross sections of the cavity but do not specify the surfaces connecting these cross sections. Although we are not able to model the exact shape of our experimental cavities the model provides insight into the many aspects of the cavity dynamics which are insensitive to small shape changes. The simplest model shape which shares the qualitative characteristics of our deformed microspheres is a spheroid with quadrupole deformation in the y - z plane and a combination of quadrupole and dipole

deformations in the x - y plane. The surface of such a spheroid can be described by the set of parametric equations

$$\begin{aligned} x &= [1 + d \cos(u) \sin(v)] [1 - q \cos(2u)] [1 + e \cos(2v)] \cos(u) \sin(v) / (1 + q) \\ y &= [1 + d \cos(u) \sin(v)] [1 - q \cos(2u)] [1 + e \cos(2v)] \sin(u) \sin(v) / (1 + q) \\ z &= [1 + e \cos(2v)] \cos(v) \end{aligned} \quad (5.3)$$

where $0 \leq u < 2\pi$ and $0 \leq v < \pi$. The quadrupolar deformation in the y - z plane (controlled by e) is the largest shape perturbation and dominates the dynamics of the ray trajectories of interest which lie near the y - z plane. The quadrupolar deformation in the x - y plane (controlled by q) is much smaller, and is oriented such that the y axis is stretched stabilizing trajectories near the y - z plane. The dipole deformation in the x - y plane (controlled by d) breaks the rotational symmetry around the z axis, simulating the effect of the stem on the spheroid's shape. The dominant effect of a dipole deformation corresponds to a shift of the center. Incorporating a simple dipole term in the model shape would produce a cusp at the poles $v = 0, \pi$ due to this shift. This cusp is removed by using $d \sin(v)$ as the dipole perturbation parameter so that the dipole deformation goes to zero at the poles. Because the size of the dipole shift is now a function of v the dipole term introduces deformations in both the x - y and x - z planes. These deformations correspond well with the expected qualitative perturbation produced by the stem which should affect both planes and yet have small effects towards the poles. When describing a position on

the surface of the resonator it is convenient to use spherical polar coordinates where the radial position is not necessary. It is important to recognize that although they are similar, the parameters u and v are not identical to the coordinates θ and ϕ .

Once the model shape is determined, rays can be iterated through successive reflections in order to calculate trajectories. The iterative process consists of propagating a ray in a straight line until it intersects the boundary surface whereupon it undergoes specular reflection determining the new propagation direction. The ray tracing code and a more detailed explanation of its operation are presented in appendix A. As in the 2D case we wish to display trajectories in a format that illuminates the patterns of the dynamics in the resonator. In the 2D case this is achieved by creating a Poincaré surface of section (SOS). The SOS for the 2D resonator reduces the 4D phase space to a manageable 2D representation. In the 2D case it is produced by recording the position and angle of incidence of each reflection off the resonator surface. We can use a similar approach in a 3D resonator. Again we choose to record the phase space coordinates of the trajectory for each reflection at the surface. In contrast to the 2D case, however, this only reduces the dimensionality from 6D to 4D. The coordinates of the surface of section would consist of the position of the reflection (θ and ϕ) and the two angles required to describe the propagation direction of the ray. Since it is not possible to present a set of points in 4D space we must find a way to reduce the dimensionality further. One way of representing the propagation direction of the ray is to specify the angle of incidence of the ray (measured from the normal to the surface) and an additional angle which defines the orientation of the plane of incidence which contains the normal and the ray. The

second angle in this representation is complicated to determine in general but fortunately is not particularly interesting either. Therefore we can make a projection of the data into a 3D space with coordinates corresponding to the position of the reflection and the angle of incidence. This modified SOS is then directly analogous to that of the 2D SOS with the exception that a single recorded point does not uniquely determine a trajectory. A further consequence of this is that, unlike the 2D SOS, two different trajectories can share a common point in the SOS. In other words, trajectories in the 3D SOS can appear to intersect one another although they do not in fact intersect in the complete 4D SOS.

Phase Space Transport in 3D Ray Model

As a first step in understanding the ray dynamics of a 3D resonator we consider the rotationally symmetric case. Using the shape described by equation (5.3) and setting $d = q = 0$ and $e = 0.06$ we examined trajectories in an axisymmetric quadrupole. Due to the rotational symmetry, the motion of a trajectory around the z axis is separable and this case can be treated as a 2D centrifugal billiard and therefore we do not expect Arnold diffusion to occur. Figure 34 shows twelve trajectories launched from the position $\theta = \pi/2$, $\phi = \pi/2$ with different initial angles of incidence but the same conserved value of L_z near zero. Each trajectory consists of approximately 200 data points which have been projected onto three planes for easier viewing. It is clear that the $\theta - \sin \chi$ projection closely resembles the standard 2D SOS for a quadrupole of the same

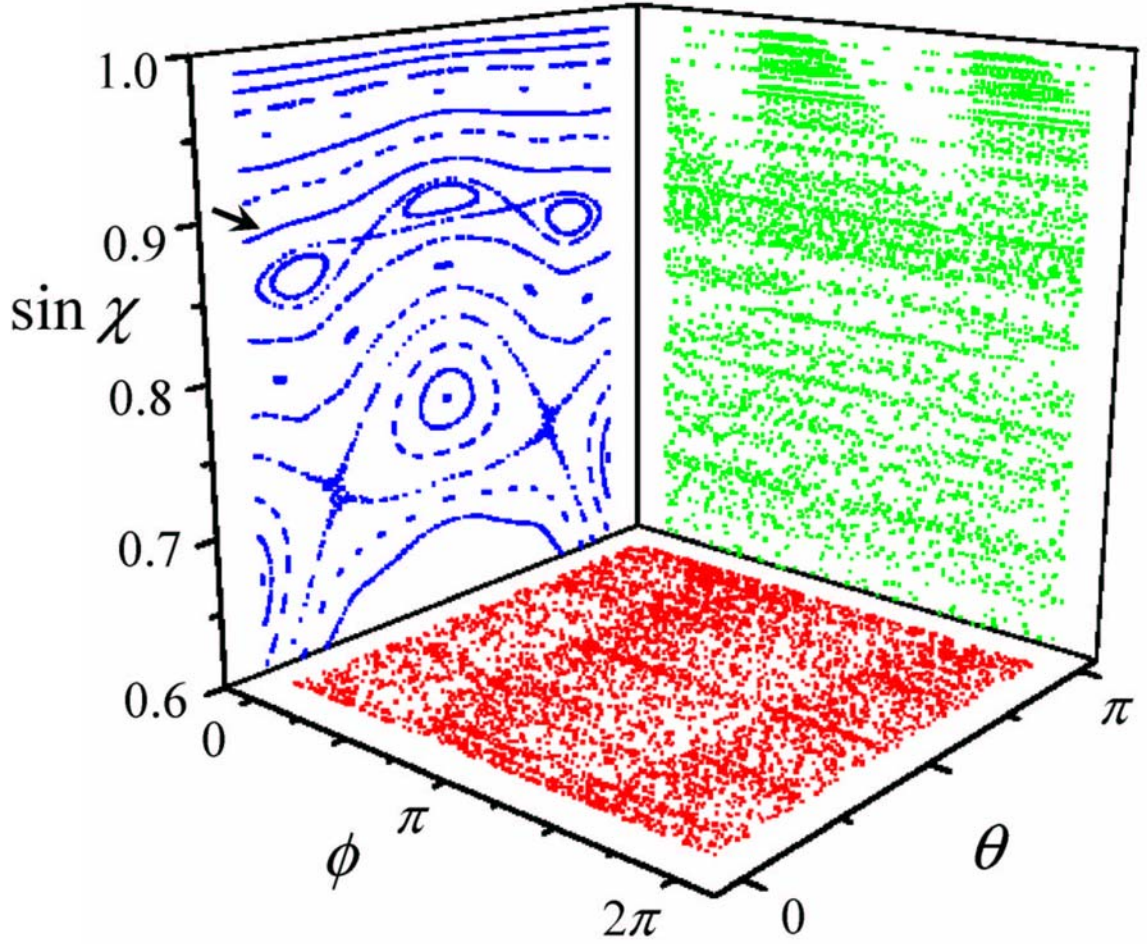


FIGURE 34. Poincaré surface of section for a rotationally symmetric 3D quadrupole. The horizontal structure in the $\phi - \sin \chi$ plane indicates that trajectories are freely precessing around the z axis.

eccentricity with stable islands, KAM tori, and chaotic separatrix regions. In fact, the $L_z = 0$ case exactly reduces to a 2D quadrupole since trajectories do not precess. The precession of trajectories when $L_z \neq 0$ can be seen by the fact that the $\phi - \sin \chi$ and $\theta - \phi$ projections are filled fairly evenly. The gaps at the top of the $\phi - \sin \chi$ plane are a

result of the fact that trajectories with large $\sin \chi$ undergo many reflections in a single orbit and thus undergo less total precession compared to smaller $\sin \chi$ trajectories when followed for the same number of reflections. All of the trajectories shown in fig 34 can be followed for an arbitrary number of reflections (the limit set by machine precision is much longer than we ever reach) with no change in appearance beyond increasing the density of points just as is the case in a 2D system. Specifically, the maximum extent of chaotic trajectories is sharply defined by surrounding KAM tori. The importance of this result is that it proves that long range diffusion in phase space is not simply the result of three dimensional motion or numerical artifacts of the computer model such as rounding.

Global diffusion of trajectories is observed when the rotational symmetry of the resonator is broken. Figure 35 shows portions of a single chaotic trajectory followed for 600,000 reflections in a nonaxisymmetric cavity described by $e = 0.06$, $d = 0.05$, and $q = 0.005$. The launch condition is identical to that of the trajectory marked with an arrow in figure 32 with $\sin \chi_0 \approx 0.9$. Each frame of figure 33 shows 5000 reflections of the trajectory at a different stage of its evolution. One immediately apparent difference from the axisymmetric case is that the x - y plane deformation has now stabilized trajectories near the y - z plane as experimentally observed. This can be seen by the fact that the trajectory does not cover the $\theta - \phi$ plane but instead is localized to $\phi \approx \pi/2, 3\pi/2$. Rather than precessing as in the axisymmetric resonator, trajectories launched near the y - z plane demonstrate a wobbling motion about the y - z plane. This wobbling motion violates the conservation of L_z and is a direct result of the removal of rotational symmetry.

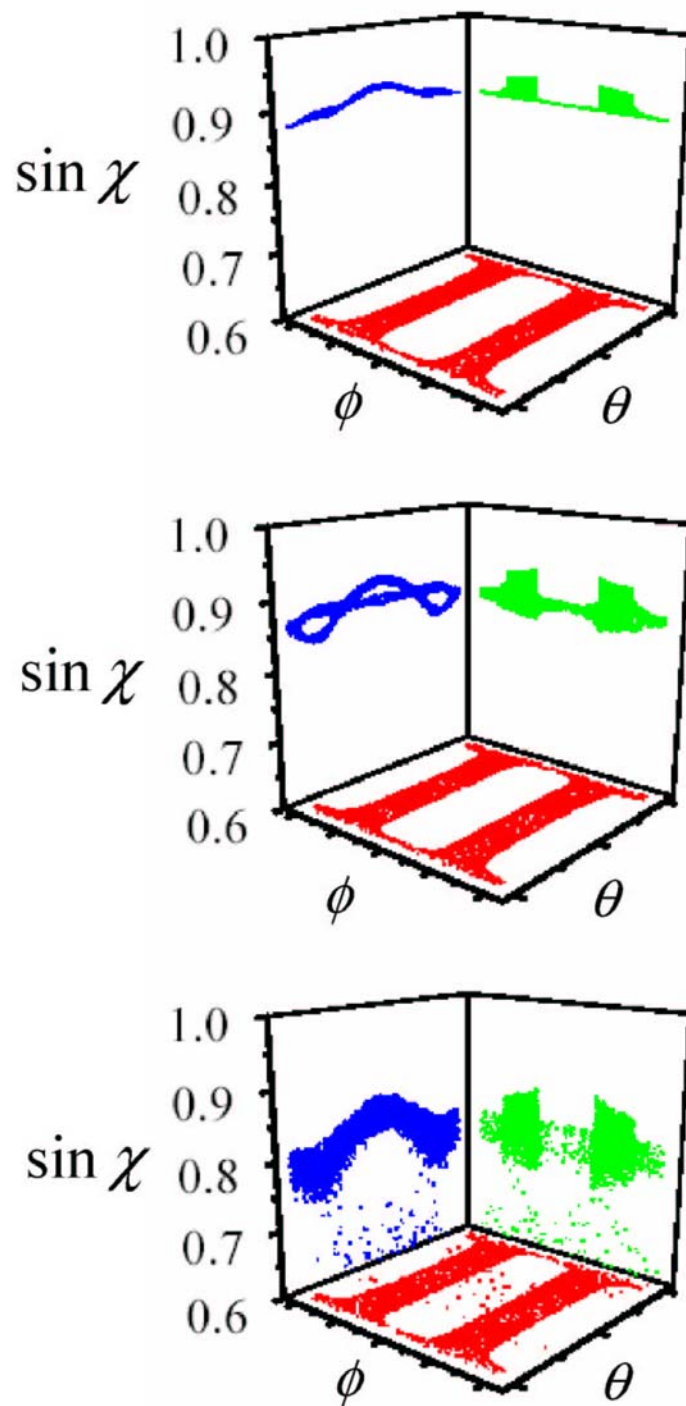


FIGURE 35. Poincaré surface of section for a nonaxisymmetric resonator. A single trajectory was followed for 600,000 reflections. From top to bottom, each frame shows 5000 reflections starting at the 1st, 310,000th and 582,500th reflection.

The qualitative behavior of the trajectory shown in figure 33 can be roughly divided into three time scales. With the weak perturbation applied to the x-y plane, at short time scales (<100 reflections) a trajectory can reasonably be considered as moving in a single specific 2D resonator. Over short time scales L_z is approximately conserved and a chaotic trajectory is therefore still effectively confined between nearby KAM tori. Over intermediate time scales (~ 5000 reflections) the wobbling motion of the trajectory becomes apparent, however, the trajectory still closely follows the pattern of a single trajectory in the corresponding rotationally symmetric resonator. The L_z nonconserving wobbling motion acts to slightly modify the exact shape of the effective 2D resonator which the trajectory experiences over short time scales. This change in effective 2D shape in turn modifies the confining KAM tori allowing the trajectory to explore a larger region of phase space than possible in the axisymmetric resonator. Over longer time scales ($>50,000$ reflections) the trajectory slowly evolves to follow different trajectories from the 2D SOS with a bias towards reducing $\sin \chi$. This diffusion toward smaller $\sin \chi$ is not a case where the intervening KAM tori have been destroyed by the deformation. In fact, many nonchaotic trajectories can be found in the range $0.7 < \sin \chi_0 < 0.9$ which show no deviation from a single surface in the 3D SOS and correspond to a single trajectory in the axisymmetric case. Rather than destroying the KAM tori, the truly 3D motion in the nonaxisymmetric resonator allows trajectories to pass around them. One can imagine a trajectory which starts in a particular plane between two KAM tori, wobbles out of that plane and later wobbles back to that plane

but now is no longer between the same two KAM tori. Continuation of this type of process can allow a chaotic trajectory to eventually approach any arbitrary point in phase space (although certainly not with equal probability).

In summary, the trajectory examined in fig 35 demonstrates two distinct types of chaotic diffusion. First, the chaotic motion within a single separatrix layer, familiar from 2D billiards, dominates at short time scales but is strongly confined in phase space by bounding KAM tori. Second, a much slower diffusion from one separatrix layer to another is observed to transport trajectories around KAM tori and provide global diffusion through phase space. This second diffusion process depends on the nonconservation of L_z resulting from the removal of rotational symmetry, and appears to be Arnold diffusion.

CHAPTER VI

WAVE DYNAMICS IN DEFORMED FUSED-SILICA MICROSPHERES

As demonstrated in Chapter V, ray tracing models predict universal far-field emission patterns from asymmetric resonators. Even in special cases, such as the dynamical eclipsing observed in deformed microspheres, the expected emission pattern is determined by the general type of deformation rather than the exact shape of the resonator. Our experimental measurements confirm this universal behavior in that deformed microspheres with similar eccentricity demonstrate qualitatively the same far field emission patterns despite differences in size and even symmetry. Although the data presented here is obtained from microspheres with approximately quadrupolar cross sections in the y-z plane, other more egg shaped microspheres produce similar emission patterns. The only exception is that deformed microspheres which maintain rotational symmetry do not demonstrate directional emission at all.

As useful as the ray tracing model is for understanding the basic dynamics of deformed fused-silica microspheres, however, the WG modes that are excited in these resonators are fundamentally wave phenomena. In this chapter we will present experimental evidence of “non-classical” behavior in deformed microspheres with large size parameters. We report directional emission from weakly deformed microspheres with an emission pattern that was not previously predicted for asymmetric cavities. We show how this emission pattern can be understood as tunneling escape directed by the

underlying ray dynamics of the cavity. We explore this interplay between ray and wave dynamics further by considering the impact that the wave nature of the deformed microsphere system has on the phase space transport. Of particular interest is what effect Arnold diffusion can have on a wave packet that cannot, in principle, resolve the infinitely fine structure of the Arnold web.

Experimental Results

In Chapter V we examined the emission pattern for a deformed fused-silica microsphere. In figure 36, the far-field emission pattern of another microsphere is presented at three different eccentricities. The average diameter of the microsphere remained about 200 microns. In each case, WG modes were launched such that $\sin \chi_0 \approx 1$. For the largest deformation ($\epsilon = 0.067$) the strong emission peak at $\theta = 45^\circ$ is well explained by dynamical eclipsing. Careful inspection of figure 36a reveals that there is also a much smaller peak (5% of the large peak height) in the far field emission pattern at $\theta = 135^\circ$. As the eccentricity of the sphere is reduced [$\epsilon = 0.022$ fig. 36b and $\epsilon = 0.012$ fig.36c] the $\theta = 135^\circ$ peak grows until it is nearly as large as the $\theta = 45^\circ$ peak. We refer to the emission pattern in figure 36c as symmetric, as opposed to the asymmetric emission that is observed at larger deformations. The emission at $\theta = 135^\circ$ corresponds to tangential escape from the resonator near $\theta = 45^\circ$ and is very interesting because it cannot be explained by refractive escape in the geometrical optics model.

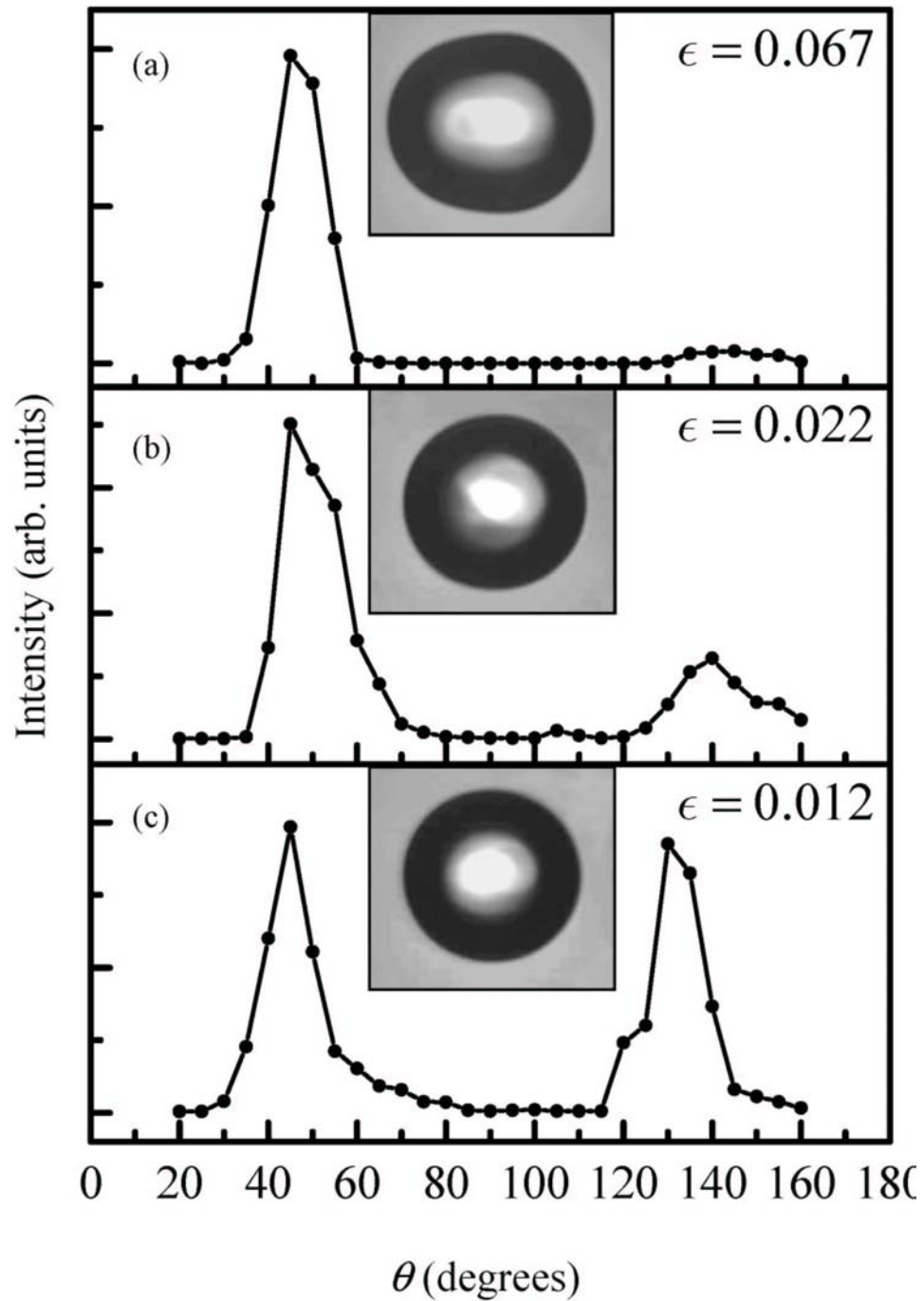


FIGURE 36. Directional emission from a deformed microsphere at three different deformations. The emission pattern makes a transition from asymmetric at large deformation to symmetric as small deformation. The insets display images of the sphere at each deformation.

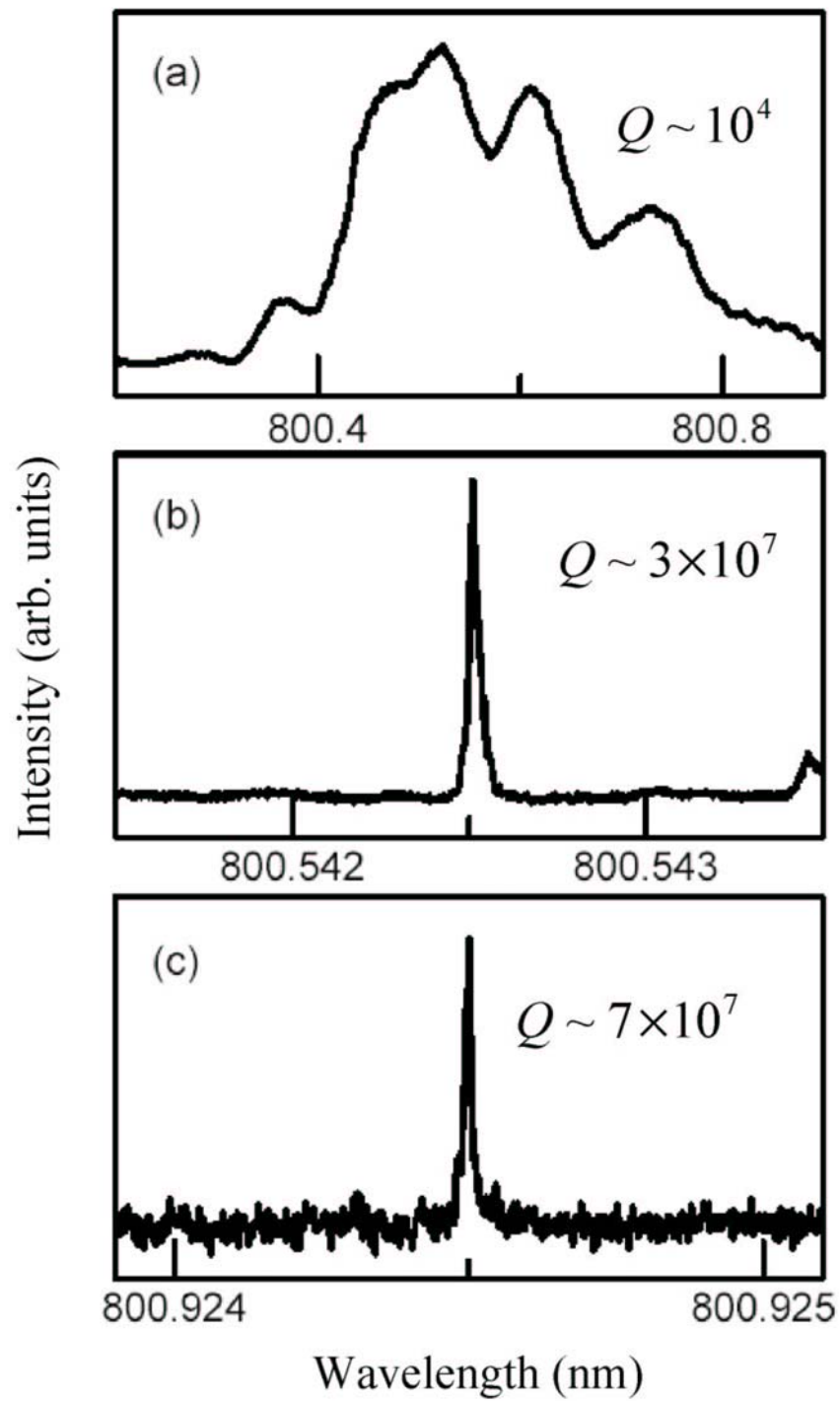


FIGURE 37. Whispering gallery resonances in a microsphere at three different deformations. The deformations correspond to the emission data in figure 34. (a) $\epsilon = 0.067$ (b) $\epsilon = 0.022$ (c) $\epsilon = 0.012$

As with the deformed microsphere examined in the last chapter, the Q-factors of the WG modes in this microsphere increased significantly as the eccentricity was reduced. Figure 37 shows the spectra and measured Q-factors corresponding to the three emission patterns of figure 36. In this case the Q-factors increased by nearly four orders of magnitude. At the smallest deformation the Q-factor reached 7×10^7 while still demonstrating highly directional emission. The high Q-factors measured for the cases where the symmetric emission pattern is observed are a further indication that refractive escape may not be the dominant emission mechanism in weakly deformed microspheres.

As a further test of our understanding of the emission dynamics in weakly deformed microspheres we repeated the measurements on the $\epsilon = 1.2\%$ microsphere with light launched closer to the critical angle. Figure 38 compares the far field emission pattern and Q factor measured at $\epsilon = 1.2\%$ for the two cases, $\sin \chi_0 \approx 1$ and $\sin \chi_0 \approx 0.8$. The far field emission pattern is essentially identical for the two cases. The Q-factors reported are measured for the WG modes that dominate the spectrum for each launch angle. The insensitivity to initial condition is confirmation that the emission pattern is determined entirely by the short term dynamics near the critical angle immediately preceding escape. As we saw in the more deformed microsphere of figure 28, the decrease in Q-factor when the initial angle of incidence is reduced indicates that the resonance lifetime is dominated by the time required for the rays to chaotically diffuse from $\sin \chi_0 \approx 1$ to a value that allows escape. In this less deformed microsphere, higher Q-factors are observed at smaller values of $\sin \chi_0$. In this case we can conclude that

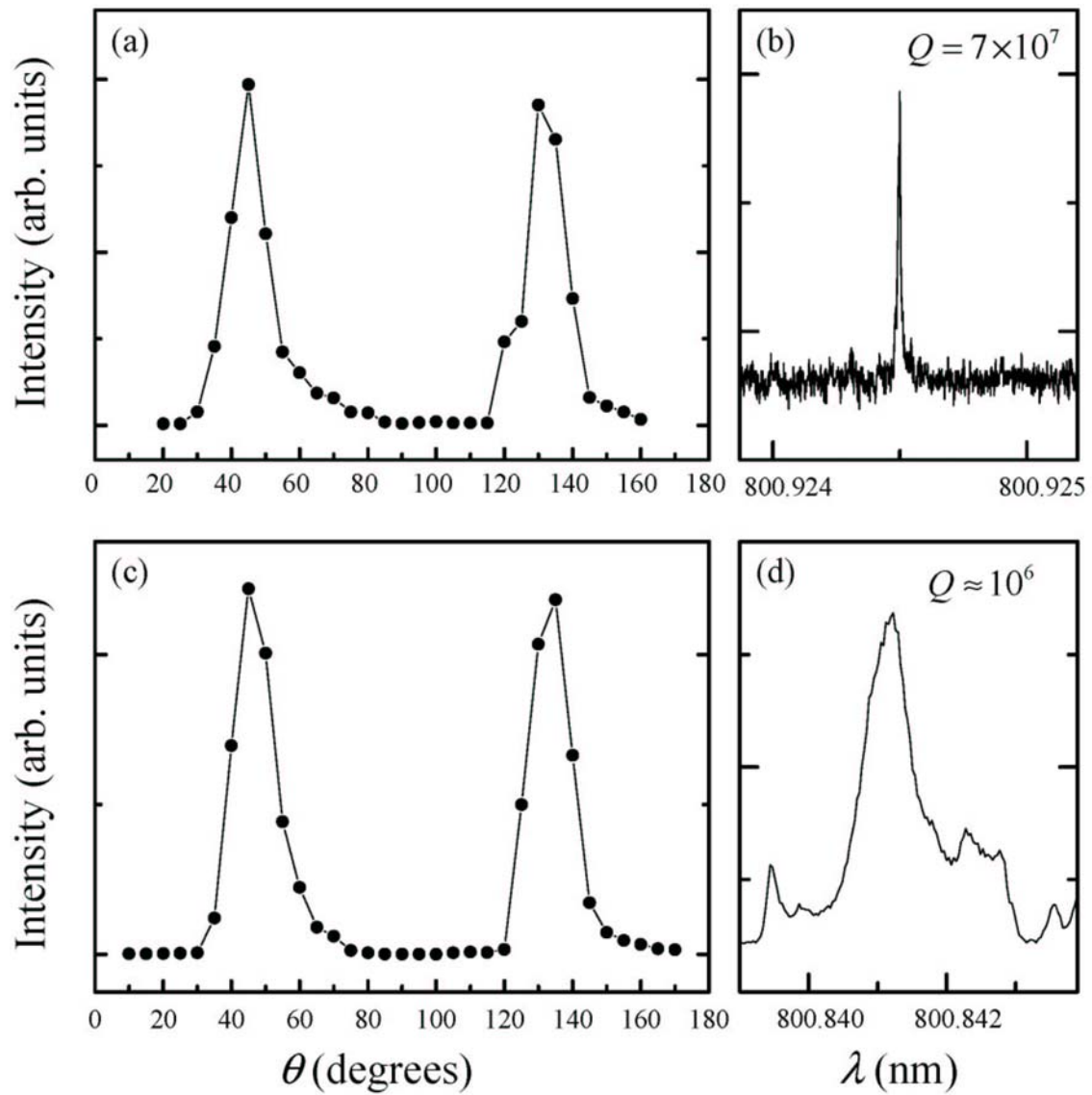


FIGURE 38. Far-field emission pattern and corresponding Q-factor for WG modes in a single deformed sphere with two different initial condition $\sin \chi_0 = 1$ (a,b) and $\sin \chi_0 = 0.8$ (c,d).

when $\sin \chi_0 \approx 1$ only 2% of a ray's lifetime is associated with diffusion from $\sin \chi = 0.8$ to the critical angle. Although the trend of increasing lifetime with decreasing deformation is consistent with the ray model, it is very unlikely that a microsphere with approximately 1% deformation is chaotic enough to explain the measured lifetimes of these modes. This observation raises the question of whether wave effects might increase the phase space transport rate compared to the ray model. This is an open question, which deserves further investigation and will be discussed further in the final chapter.

As the initial angle of incidence was reduced even further the resonance spectrum became sensitive to where on the resonator surface the light was input, θ_0 , as well as $\sin \chi$. All of the previous measurements were made with the long axis of the deformed microsphere parallel to prism face. Figure 39a shows the resonance spectrum measured for $\sin \chi_0 = 0.73$ in this configuration where $\theta_0 = -90^\circ$. With these initial conditions the dominant modes that are excited have $Q \sim 10^3$. Much more weakly excited modes with Q-factors as high as 10^6 can be seen on top of the dominant low Q modes. For figure 39b, the microsphere was rotated relative to the prism as indicated by the diagram. With the input conditions $\sin \chi_0 = 0.73$ and $\theta_0 = -135^\circ$ the dominant modes in the spectrum have $Q \sim 10^6$.

The transition from asymmetric to symmetric emission pattern is gradual as the eccentricity is reduced. Over the same range of deformations a transition is also observed in the resonance spectrum of the microsphere. We have already discussed the large

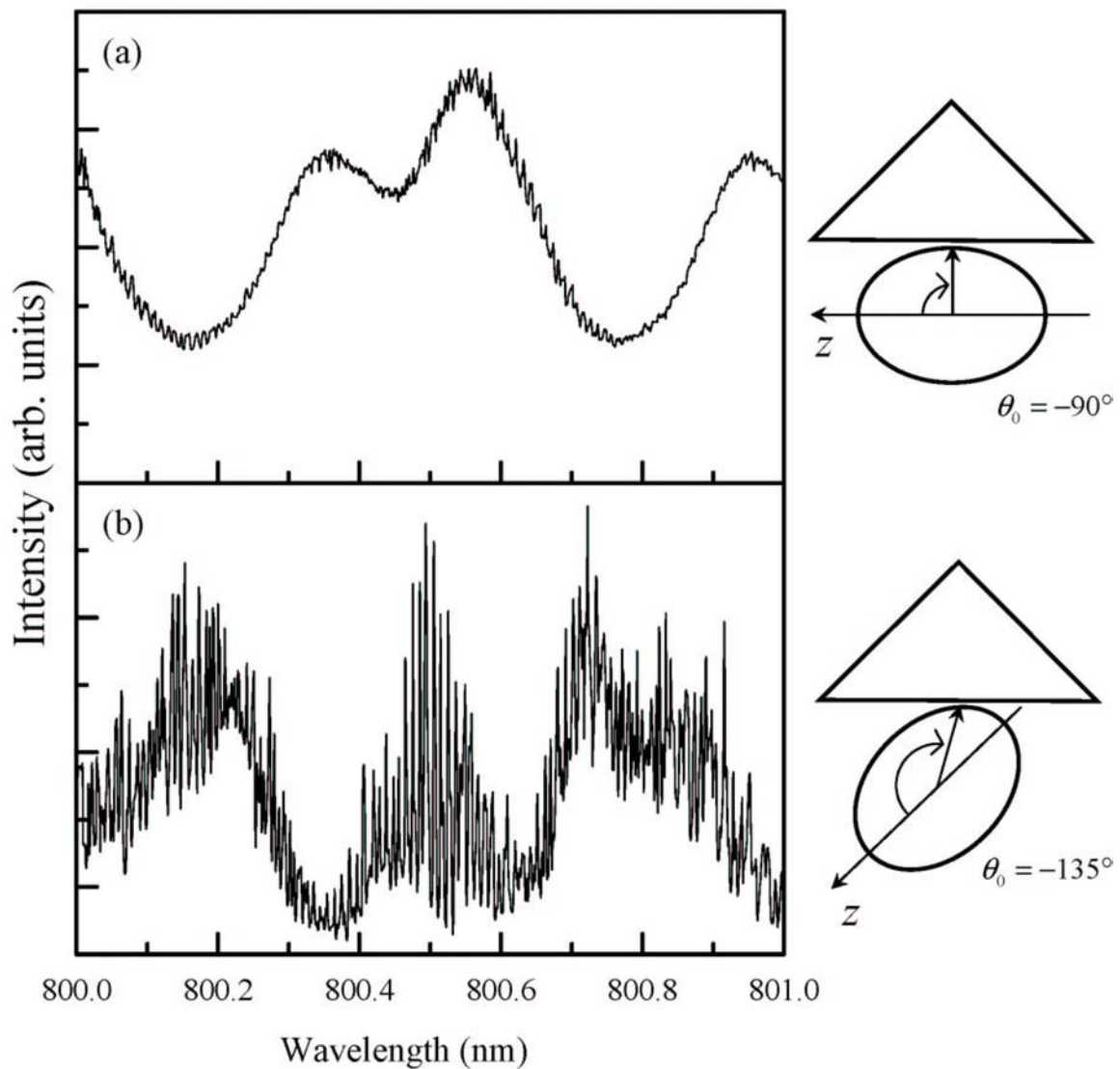


FIGURE 39. Resonance spectra of a deformed microsphere at $\sin \chi = 0.73$. (a) When the long axis of the microsphere is parallel to the prism face the excited mode have very low Q-factors ($Q \sim 10^3$). (b) When the microsphere is rotated 45° modes with $Q > 10^6$ are excited much more strongly. The size and deformation of the microsphere are exaggerated in the diagrams to indicate its orientation.

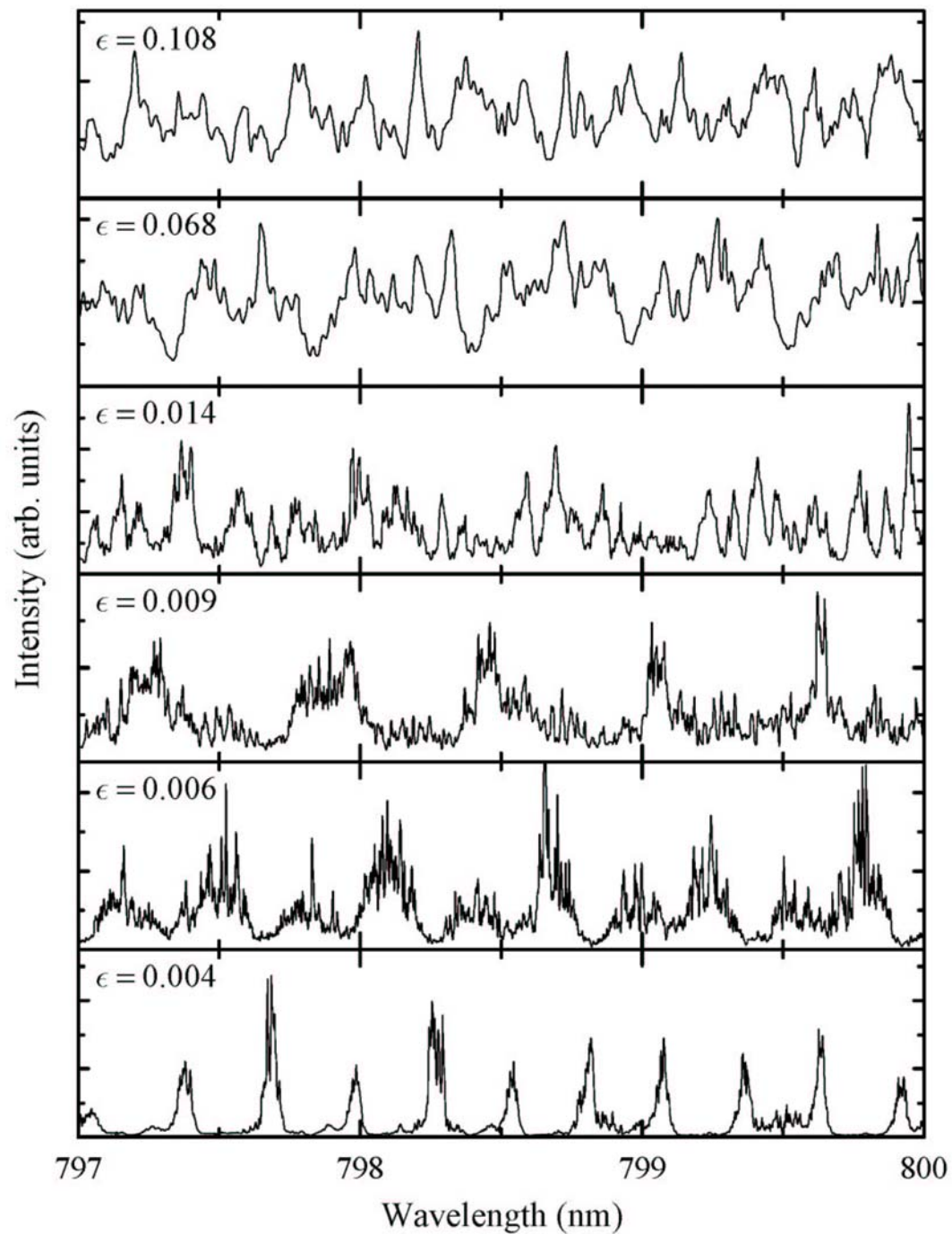


FIGURE 40. Whispering gallery mode spectra of a single deformed microsphere at six different deformations. The free spectral range of 0.55 nm is only clearly visible in the less deformed spheres. Directional emission was observed in all but the $\epsilon = 0.004$ sphere.

change in resonance linewidth, which is concurrent with the change in emission directionality. In addition the large scale structure of the resonance frequencies also changes. Figure 40 shows a sequence of resonance spectra obtained at different eccentricities. In strongly deformed microspheres the spectrum is quite disordered. The free spectral range is the most robust feature of the spectra, but even this is not always easily discernable at large eccentricity. At smaller deformations the periodic structure that repeats every free spectral range becomes more apparent (fig 40d-f). At $\epsilon = 0.004$ the resonance spectrum looks very much like that of a standard nondeformed microsphere. The major difference in appearance of the spectra in the range $0\% < \epsilon < 1\%$ can be understood qualitatively as the splitting between modes of different m increasing with the deformation. The similarity between the WG mode spectra for the weakly deformed microspheres and a nondeformed microsphere suggests that perturbation theory could be employed to determine the resonance frequencies in the deformed case. It is interesting to note, however, that the emission pattern starts to become symmetric before the spectrum develops the periodic pattern resembling the nondeformed case. Furthermore, in the following section we will demonstrate the observed emission directionality is produced by non-perturbative effects even in the weakly deformed microspheres.

Directional Tunneling from Weakly Deformed Microspheres

The far-field emission pattern that is observed from weakly deformed microspheres cannot be explained by either the 2D or 3D ray tracing models. Neither of these models can account for the light that escapes the resonator tangentially from the point $\theta = 45^\circ$. In order for a ray to arrive at $\theta = 45^\circ$ with $\sin \chi < \sin \chi_c$ it must have already undergone numerous previous reflections also with $\sin \chi < \sin \chi_c$. This is because trajectories must circulate counterclockwise around the islands. A ray would, therefore, have already escaped the cavity prior to arriving at $\theta = 45^\circ$. Although the four bounce island structure in the SOS has the same symmetry as the emission pattern observed at small eccentricities, the flow of trajectories around the islands produces an asymmetric emission pattern matching what is observed for large eccentricities. This leads us to the conclusion that the symmetric emission pattern cannot be associated with refractive escape of trajectories circulating around the stable islands.

We attribute the symmetric emission pattern observed at small eccentricities to directional tunneling escape. This is surprising given the large size parameters being considered, however, we will show that in certain conditions the tunneling rate for light escaping a microsphere can be quite large despite large size parameters. In an ideal sphere refractive escape of WG modes is not possible because the angle of incidence does not change. Tunneling, or evanescent leakage, is also negligible for small ν modes ($\sin \chi \approx 1$) in a nondeformed sphere. The tunneling escape rate increases exponentially,

however, as $\sin \chi$ approaches $\sin \chi_c$. For instance, in a $100 \mu\text{m}$ radius fused-silica sphere at $\lambda = 800 \text{ nm}$ the tunneling limited Q factor drops from 10^7 to 10^5 in the range $0.71 < \sin \chi < 0.70$. An approximate expression for the Q-factor of WG modes in a cylinder has been derived⁴²,

$$Q \approx \frac{n\pi(ka)^2 [J_l^2(ka) + Y_l^2(ka)]}{\ln\left(\frac{n+1}{n-1}\right)}. \quad (5.1)$$

Based on our geometrical optics interpretation that for large size parameters $l \approx nkR \sin \chi$, we can calculate Q factors as a function of $\sin \chi$. Figure 41 shows the tunneling limited Q-factor for WG modes near the critical angle for spheres with $100 \mu\text{m}$ and $50 \mu\text{m}$ radii. Because of the strong dependence on $\sin \chi$, under certain conditions it is possible that tunneling escape of rays with $\sin \chi < \sin \chi_c$ will be the dominant emission mechanism for WG modes of deformed microspheres. In order for tunneling escape to dominate, the light in a WG mode must spend adequate time with $\sin \chi$ only slightly greater than $\sin \chi_c$. This condition is consistent with our observation that as the eccentricity of the microsphere was reduced and the emission pattern became more symmetric, the Q factor of the modes increased by nearly four orders of magnitude. The relatively low Q of the strongly deformed microsphere indicates that the resonance lifetime is dominated by refractive escape. For the smaller deformations the Q approaches that of a nondeformed microsphere indicating that tunneling escape has become the dominant emission mechanism.

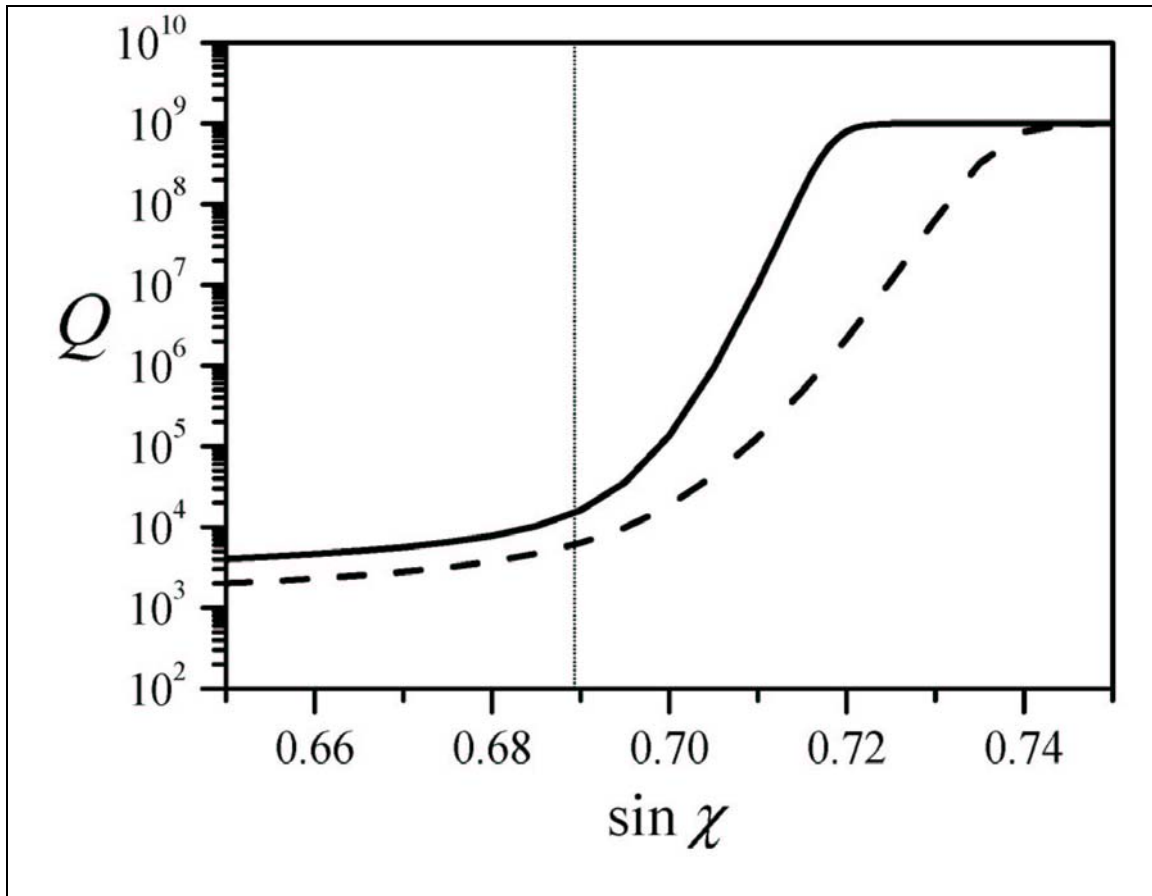


FIGURE 41. Calculated Q-factor versus $\sin \chi$ in a nondeformed sphere of radius 100 mm (solid) and 50 mm (dashed). The calculation assumes a material limited Q-factor of 10^9 . The dotted line indicates the critical angle for total internal reflection.

Interestingly, once it is recognized that the emission from weakly deformed microspheres must be due to tunneling, the ray model can again be used to explain the direction of the emission. As discussed previously, over short time scales ray trajectories in a 3D resonator tend to follow phase space structures seen in the corresponding 2D SOS. Therefore the 2D SOS is a useful simplification of the full 3D ray model and is justified when considering the short term behavior immediately prior to escape from a 3D cavity. For any single round trip of a ray, the cross section

of the cavity which the ray encounters can be well approximated by a 2D boundary. For successive round trips this 2D cross section changes producing a slight change in eccentricity experienced by the ray. These small changes in effective eccentricity have negligible effect on the size of the stable islands and therefore should have negligible effect on the emission pattern. When a ray is following a KAM curve one can write a function $\chi(\theta)$ that specifies the angle of incidence as a function of position on the resonator surface. Because of the exponential dependence of the tunneling escape rate on χ , tunneling escape will occur predominantly at the minima of $\chi(\theta)$. Inspection of the 2D SOS for a quadrupole resonator (for example figure 31a) reveals that this directional tunneling escape effect will be strongest for trajectories just above the four bounce island chain ($\sin \chi > \sin 45^\circ$). There are two main reasons which cause this region to dominate the directional tunneling escape. First, because the angle of incidence is so close to the critical angle the tunneling rate is larger than that for any other set of trajectories which do not escape refractively. Second, the four bounce island chain is the strongest periodic feature in the SOS producing the largest oscillations for $\chi(\theta)$ anywhere above the critical angle. Large oscillations in $\chi(\theta)$ produce maximum contrast between strong and weak emission directions due to the exponentially sensitive tunneling rate. For trajectories just above the four bounce islands $\chi(\theta)$ has minima near $\theta = \pm 45^\circ$. This corresponds well with the symmetric emission points observed for weakly deformed microspheres.

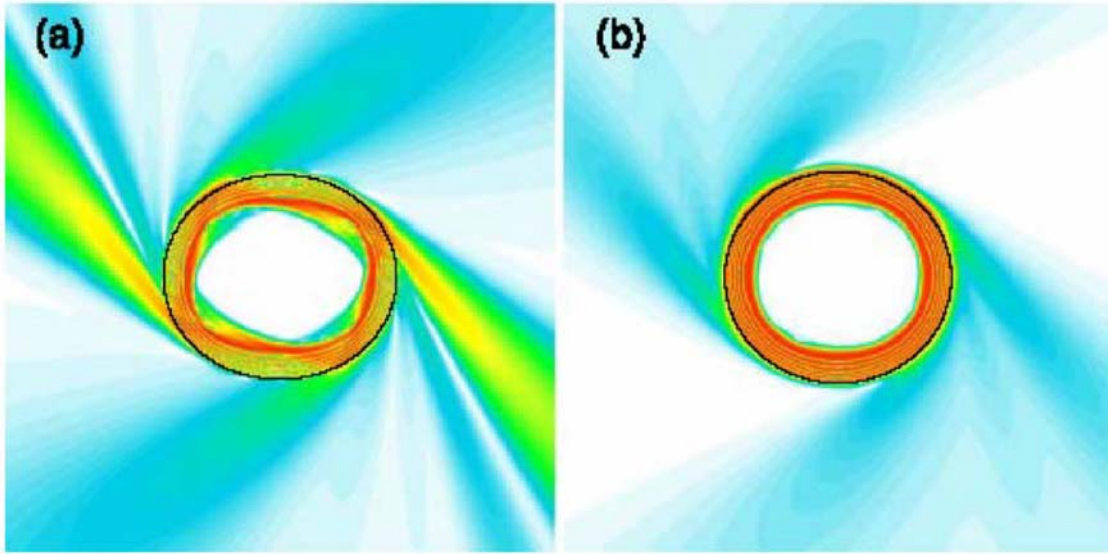


FIGURE 42. False color intensity plots of WG modes in two deformed cavities. Numerical solutions of the wave equation reproduce the measured emission patterns of (a) strongly deformed ($\epsilon = 0.065$) and (b) weakly deformed ($\epsilon = 0.034$) spheres. [figures courtesy of Dr. Jens Nöckel]

Both the asymmetric emission pattern of the strongly deformed microsphere and the symmetric emission pattern observed from weakly deformed microspheres can be reproduced in a numerical wave solution of Maxwell's equations for a 2D quadrupole. Figure 42 shows false color intensity contour maps of resonances in quadrupoles with $\epsilon = 0.065$ and $\epsilon = 0.034$ closely corresponding to the observed emission patterns of figure 36a and 36b respectively. Similarly to dynamical eclipsing, the emission pattern produced by directional tunneling depends on the index of refraction and the phase space structure of the resonator. For a resonator

with refractive index $n = 2$ directional tunneling could occur, however the light would escape predominantly from the points of highest curvature as it would for refractive escape with $n = 2$. The interesting feature about directional tunneling from fused-silica microspheres is that there is a qualitative difference between the refractive emission pattern and the tunneling emission pattern. This allows the emission pattern to act as a sensitive probe of the relative importance of refractive and tunneling escape, which would not be possible in the general case.

The data presented in figure 39 is another example of the interplay between ray and wave effects in a deformed microsphere. The difference in these two spectra can be understood by considering that these two different initial conditions lay on different invariant curves in the SOS despite the fact that they share the same $\sin \chi_0$. The presence of the stabilized four bounce islands causes significant curvature of the KAM curves in this region of phase space. The result is that rays following a KAM curve after being launched with the initial conditions of figure 39a will reach much lower values of $\sin \chi$ than rays launched with the initial conditions of figure 39b. This can have a very large effect on Q-factors due to the exponential dependence of the tunneling rate on $\sin \chi$ in this region of phase space. The far-field emission pattern in this case still indicates that the light escapes by tunneling. In order to explain the large difference in WG mode lifetimes for the two input conditions,

however, it is necessary to understand both the exponential dependence of the tunneling rate near the critical angle and the non-perturbative phase space structure in the vicinity of the four bounce periodic orbit.

In this chapter we have placed an emphasis on the interesting behavior that arises under conditions where both ray effects and wave effects are important. Gaining an increased understanding of the dynamics in this regime is of great interest to researchers in a variety of fields. We have shown that although a perturbative approach may be appropriate for understanding some properties of weakly deformed microspheres, non-perturbative effects can strongly influence the dynamics down to surprisingly small deformations.

CHAPTER VII

FREE-SPACE COUPLING TO DEFORMED FUSED-SILICA MICROSPHERES

To this point, our discussion of the optical properties of deformed microspheres has focused on emission directionality and resonance lifetime. An understanding of these properties is of primary importance for applications such as microlasers and cavity QED experiments. It is also important, however, to consider the practicality of getting light into the resonator. Although in these applications it is possible to derive practical benefits from the directional emission provided by deformed microspheres while relying on prism or fiber input coupling, it would be desirable to dispense with the need for the near field coupler entirely. This is especially true in sensor applications where it might be particularly inconvenient to manipulate delicate near field coupling devices in a sealed vessel. Furthermore, near field couplers are extremely sensitive to vibrations in situations where they must be slightly separated from the microsphere to maintain high Q . When a near field coupling device makes contact with a microsphere it tends to stick due to van der Waals and electrostatic forces. In this way, vibrations are not simply a source of noise when using near field couplers, but rather often render near field coupling techniques practically impossible without settling for reduced Q -factors by letting the coupler remain in contact with the microsphere.

In light of these considerations, this chapter investigates the practicality of free-space input coupling to deformed fused-silica microspheres. Having demonstrated

directional emission, it is clear that free-space input coupling of some sort must be possible due to time-reversal symmetry. It is not obvious, however, that it will be possible to excite high-Q WG modes without the use of near field couplers which allow one to start rays at large values of $\sin \chi$. We demonstrate that free-space input coupling to a weakly deformed microsphere leads to excitation of modes spanning a wide range of Q-factors. In order to understand this interesting phenomenon we also present WG mode spectra obtained by prism coupling over a range of $\sin \chi_0$. The practicality of free-space coupling is shown to be greatly improved by the presence of the stabilized four-bounce islands which prevent emission from the point of highest curvature on the microsphere.

Free-Space Coupling to Strongly Deformed Microspheres

Let us first consider the case of coupling light into a strongly deformed ($\epsilon > 0.04$) microsphere without a near field coupling device. In order to achieve free-space coupling, the strong emission directions were first determined by observing the emission of WG modes excited using the prism coupling technique. As previously established, the emission pattern from a strongly deformed microsphere consists of two strong emission directions. Once these directions were determined, the prism was removed and the laser was focused directly at the edge of the microsphere along one of the strong emission directions. The laser light is coupled into the microsphere by refraction and circulates in the microsphere in the opposite direction from when prism coupling was used. The light eventually escapes the microsphere predominantly along two directions nearly

perpendicular to the input beam. Figure 43 indicates the relevant directions and shows how free-space input coupling can be thought of as the time reversal of directional emission.

The output power observed in a single far field direction was as high as 10% of the power of the input laser beam. Because the far-field emission pattern is dominated by two roughly equal intensity peaks, we estimate that the input coupling efficiency was nearly 20%. This efficiency was optimized by adjusting the orientation between the input beam and the deformed microsphere. The impact parameter of the input laser on the microsphere was easily adjusted to optimize the output signal by translating the focusing lens. Not surprisingly, we found that the coupling efficiency was strongly sensitive to the

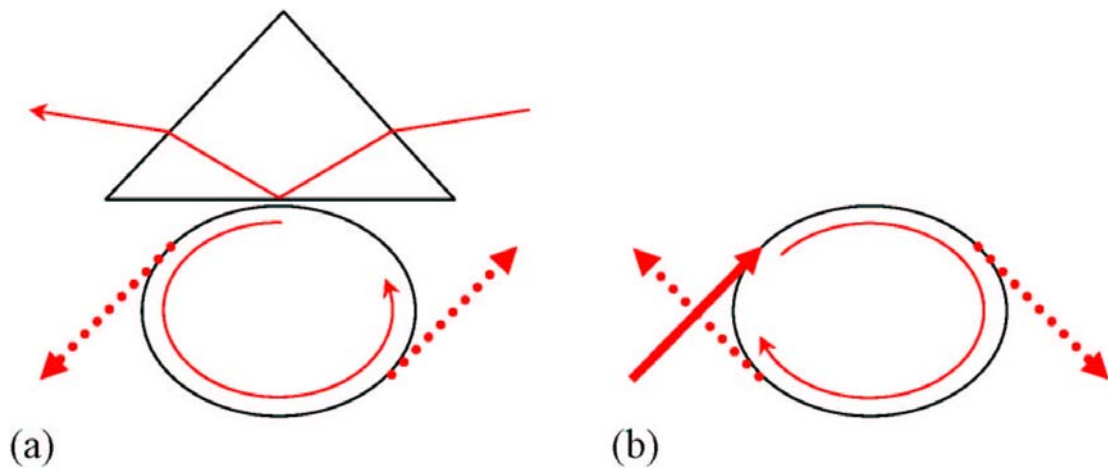


FIGURE 43. Free-space coupling to a strongly deformed microsphere. (a) Prism coupling is used to determine the strong emission directions indicated by dotted arrows. (b) The prism is removed and the laser (solid arrow) is focused at the edge of the microsphere along the strong emission direction.

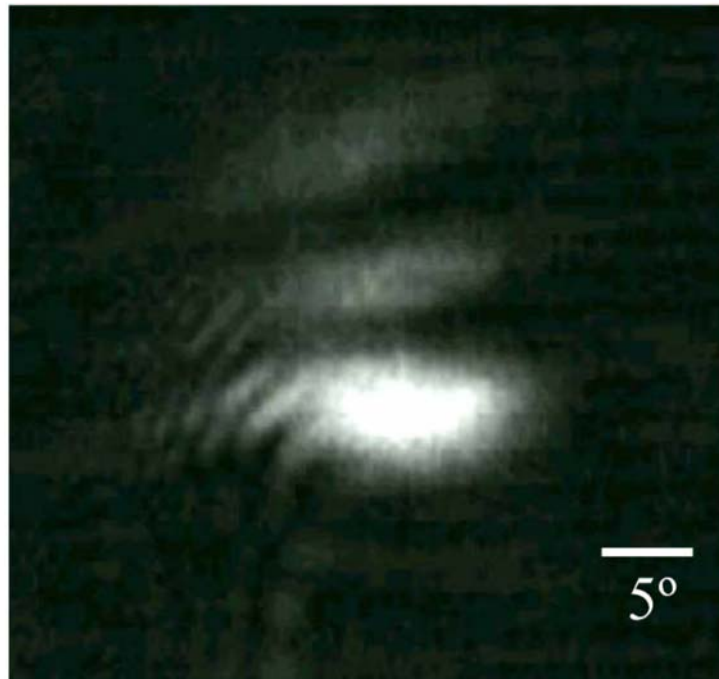


FIGURE 44. Far-field emission pattern from free-space coupled deformed microsphere.

focusing of the input beam. In addition, we found it necessary to carefully adjust the input point on the microsphere by rotating it about the stem axis. Although perhaps not the most sensitive parameter for providing maximum coupling, this was the most difficult adjustment to make since it required readjusting nearly every other parameter each time. In our demonstration of free-space coupling we used a standard 4x microscope objective lens to focus the laser onto the microsphere. We expect that further improvements to the input coupling efficiency could be made by carefully considering mode matching and input lens selection. Figure 44 shows the far field mode pattern of a free-space coupled resonance projected onto a screen. This emission pattern suggests that optimum mode matching may require the use of cylindrical lenses.

The resonance spectrum of a strongly deformed microsphere excited by free-space coupling is shown in figure 45. The Q-factor of the modes is on the order of 10^4 as expected for a strongly deformed sphere. At this low Q-factor the output should perhaps be considered to be the result of multiple beam interference rather than WG modes. Nevertheless, the high peak-to-background ratio and relatively efficient refractive coupling could make for an attractive filter in micro-optics applications.

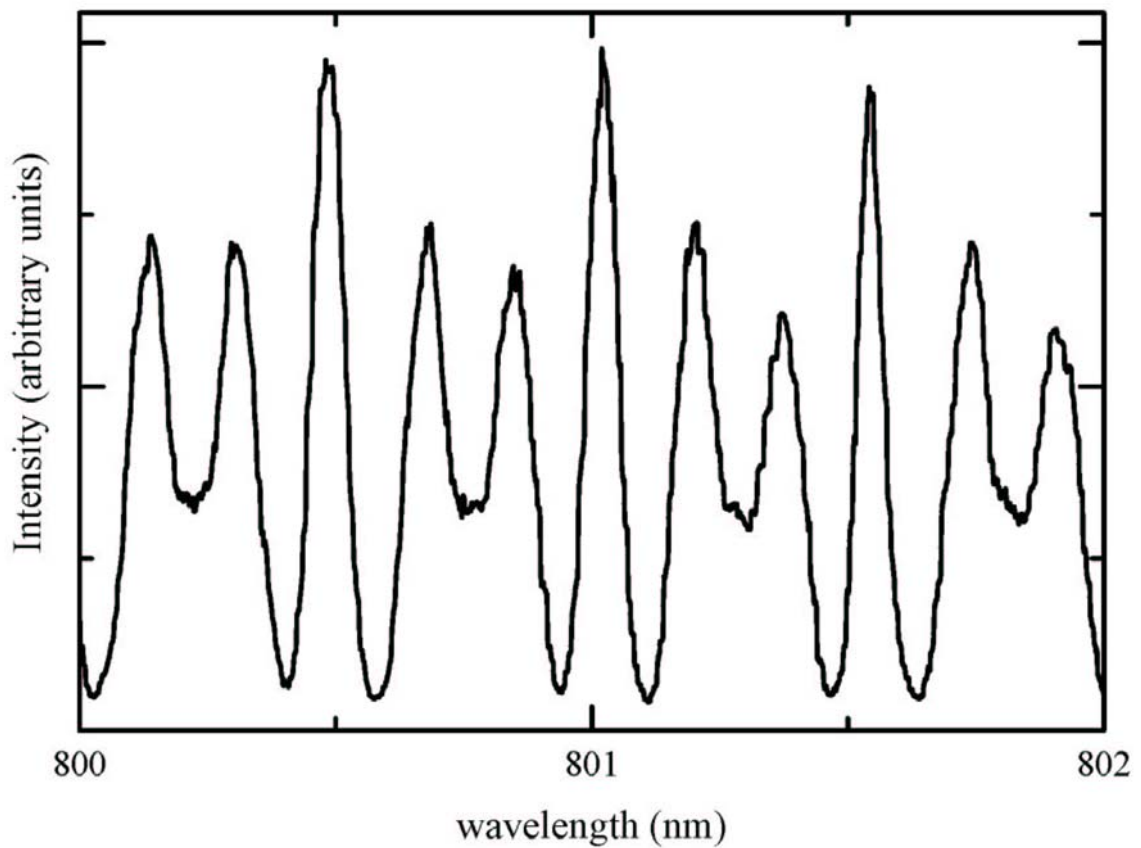


FIGURE 45. Resonance spectrum of a strongly deformed microsphere excited by free-space coupling.

Free-Space Coupling to Weakly Deformed Microspheres

In order to achieve free-space coupling to high-Q WG modes one must work with a weakly deformed microsphere ($\epsilon \sim 0.01$). In Chapter VI we demonstrated that although weakly deformed microspheres do possess highly directional emission, it is not a result of refractive escape. Therefore, we cannot expect to achieve refractive input coupling to high-Q WG modes. If free-space coupling to high-Q modes may only be accomplished via a tunneling process one might question its significance. In addition to providing a deeper understanding of the subtle dynamics of weakly deformed resonators, however, we have found that free-space coupling can provide reasonably efficient excitation of high-Q WG modes.

There are two main differences when free-space coupling to weakly, rather than strongly, deformed microspheres. First, the input coupling to high-Q modes is accomplished by tunneling rather than refraction. The focal point of the laser beam is therefore positioned just outside the microsphere. Although the coupling efficiency is less, there is an advantage in that most uncoupled light continues past the microsphere in a well defined cone rather than scattering off of the surface. The second difference is the presence of two additional strong output directions. These two additional output beams are nearly parallel and anti-parallel to the input beam, and thus are difficult to separate from the background and of little practical use. Figure 46 shows a microsphere excited by free-space coupling imaged from two different directions. For each image the orientation of the camera relative to the input beam and four output directions is indicated in the drawings.

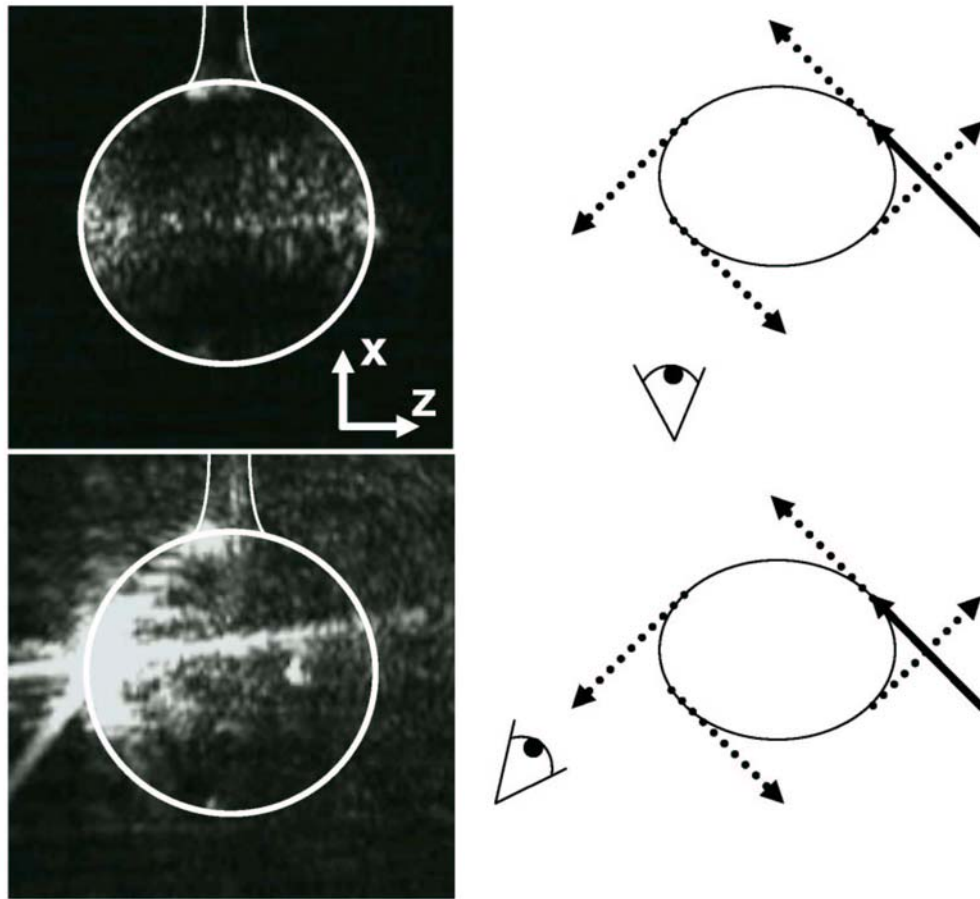


FIGURE 46. Images of a deformed fused-silica microsphere excited by free-space coupling. The outline of the microsphere is drawn to guide the eye. To the right of each image a schematic indicates the input direction (solid arrow), output directions (dashed arrows) and observation direction (eye).

The WG mode spectrum of a weakly deformed microsphere ($\epsilon = 0.01$) excited by free-space coupling is presented in figure 47. WG modes with Q-factors ranging from 5×10^4 to 4×10^7 are displayed. Similar behavior is displayed when WG modes are excited using prism coupling with $\sin \chi_0 < 1$. The decrease in Q-factor with decreasing

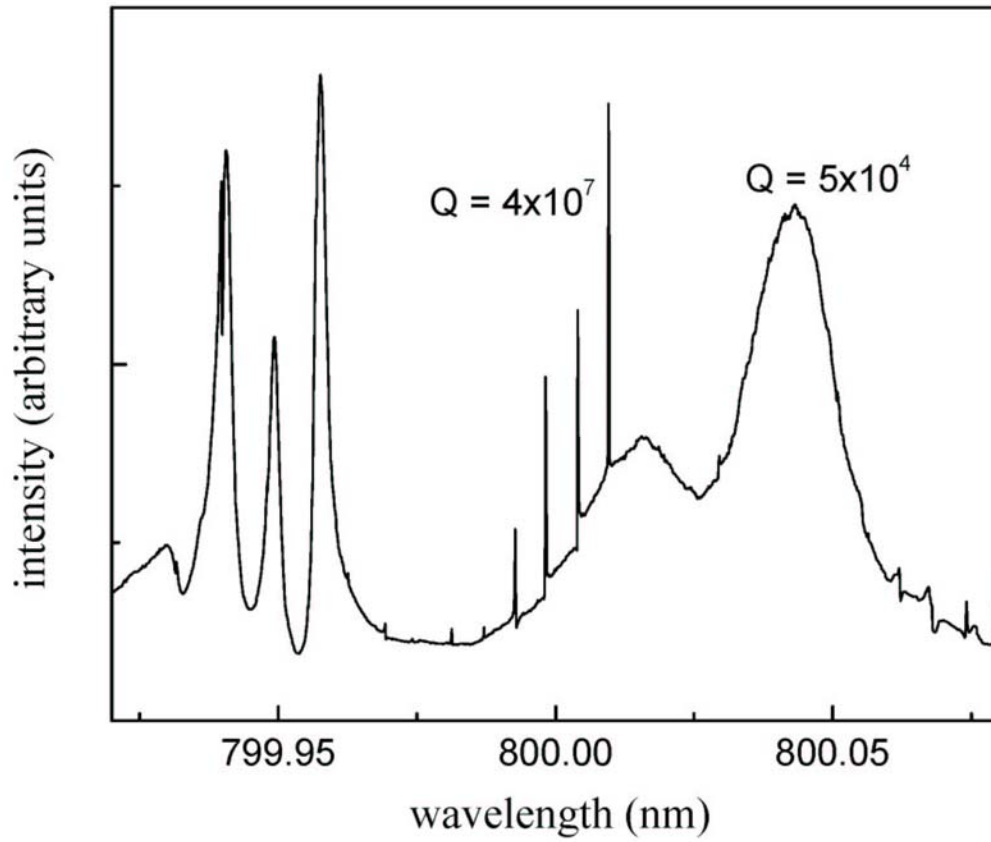


FIGURE 47. Far-field WG mode spectrum of deformed fused-silica microsphere excited by free-space coupling. Q-factors are given for representative high and low-Q modes.

$\sin \chi_0$ was discussed previously and summarized in figure 28. Figure 48 shows portions of the resonance spectra, obtained from the same microsphere at different values of $\sin \chi_0$. These spectra demonstrate that although WG modes with lower Q-factors are excited as $\sin \chi_0$ decreases, high-Q WG modes continue to be excited. Free-space input coupling can be considered as the limit of reducing $\sin \chi_0$ to $\sin \chi_c$. All of the WG

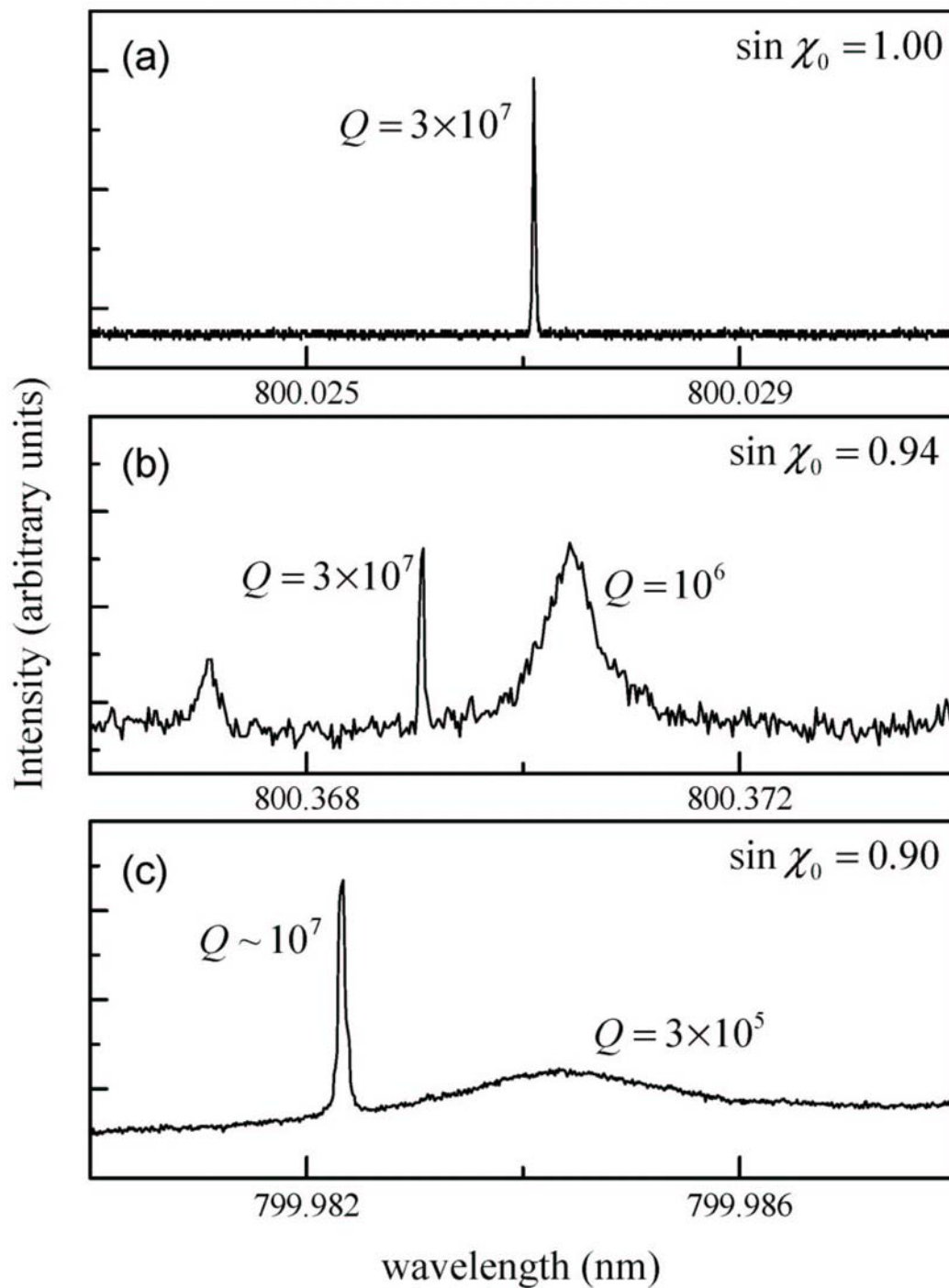


FIGURE 48. WG mode spectrum of deformed fused-silica microsphere excited by prism coupling with different initial incidence angles, χ_0 . As $\sin \chi_0$ is reduced, low-Q modes begin to be excited while high-Q modes remain.

modes measured for figures 28 and 48 exhibited nearly identical emission patterns. Free-space input coupling corresponds to the reverse of this emission process and should therefore be expected to excite all of the modes that share the same emission pattern.

Comparison of Free-Space and Prism Coupling

It is quite interesting to note that the input coupling efficiency to the high-Q modes is as good as or better than the coupling efficiency for low-Q modes in the weakly deformed microsphere. Taking into account the fact that there are four strong output directions we estimate that the input coupling efficiency to the high-Q WG modes is on the order of a few percent. This efficiency actually compares favorably with that typically obtained by prism coupling when high-Q modes are required. With the prism, it is necessary to maintain a small separation from the sphere in order to observe modes with $Q > 10^6$. Prism coupling, therefore, also suffers from reduced coupling efficiency due to the necessity to tunnel to high-Q modes.

In order to maintain high Q-factors in WG resonator a ray must have on average a small probability of escape at each reflection. Directional input/output coupling is achieved when the probability of escape for a ray is only significant for a small localized subset of the reflections that comprise its trajectory. With prism or fiber coupling to a microsphere resonator this condition is satisfied because the input/output window is localized near the point of contact with the coupler. The probability for transmission through this window is controlled by the prism-sphere gap. While the angle

of incidence for each reflection remains constant inside the symmetric resonator, the critical angle for total internal reflection is changed (eliminated, in fact) at the contact point allowing rays to escape into the coupler. In a deformed microsphere the conditions for high-Q modes with directional input/output coupling are satisfied differently. Instead of changing the condition for escape at one location, the shape of the resonator causes the angle of incidence to change. Consequently, it is possible for the majority of reflections in a ray's trajectory to be totally internally reflected despite entering and escaping the resonator with high probability at certain points. In addition, the presence of chaos in the phase space of the deformed microsphere makes it possible for a ray to change its angle of incidence such that it has very low probability of escape at any points for long periods of time. The lifetime of a trajectory in a deformed microsphere is, therefore, determined by how much time the trajectory spends with $\sin \chi > \sin \chi_c$.

In a wave model the non-conservation of the angle of incidence can be viewed as mixing of the eigenmodes of the ellipsoidal resonator. For a weakly deformed microsphere we could justifiably consider the angle of incidence conserving modes of the microsphere rather than those of an ellipsoid. As discussed previously, however, the details of the mode mixing that occurs in a nonaxisymmetric resonator which exhibits globally chaotic trajectories may be very complex. In this picture the Q-factor of a resonance in a deformed microsphere is determined by the relative weighting of the constant- χ modes that comprise it. All modes of a deformed microsphere that exhibit strong directional emission must have significant weighting near $\sin \chi \approx \sin \chi_c$. The

extent of a mode towards larger angles of incidence, however, will determine its Q-factor. The variation of this extent among different modes produces the large range of Q-factors observed in figure 47. This also explains how the Q-factor depends on the initial angle of incidence when using prism input coupling. When the initial angle of incidence is $\sin \chi \approx 1$, as in figure 48a, only modes which extend to large values of $\sin \chi$ are excited and therefore only large Q-factors are observed. As $\sin \chi_0$ is reduced these same high-Q modes can still be excited, however, it is also possible to excite modes with less extent in $\sin \chi$ and correspondingly smaller Q-factors. Free-space coupling can, again, simply be viewed as the limiting case where all directionally emitting modes can be excited.

In both the free-space coupling and prism coupling techniques the limiting Q-factor of WG modes is determined by the output coupling rate, which is directly related to the input coupling rate. For prism coupling the input/output rates are determined by the size of the gap between the prism and microsphere. With free-space coupling, on the other hand, the input/output rates are determined by the intrinsic properties of a particular WG mode. Aside from mode matching considerations, we can see that the two techniques should always provide comparable input efficiencies for modes with the same effective Q-factor.

CHAPTER VIII

SUMMARY AND FUTURE WORK

Summary

This dissertation has presented both experimental and numerical studies of deformed fused-silica microspheres. We have described the technique, which we developed to fabricate this new type of asymmetric dielectric optical resonator. Deformed microspheres provide an excellent system for studying asymmetric resonator dynamics because they are a solid-state, passive optical resonator. The excellent optical properties of fused-silica allow one to study extremely long lived chaotic WG modes that cannot be observed in asymmetric resonators made from other materials. Deformed microspheres also provide the advantage of being a solid-state cavity where the deformation may be adjusted to study a single resonator across a range of deformations. Finally, deformed microspheres provide the unique opportunity to study a three dimensional asymmetric resonator, for which the internal ray dynamics are not reducible to a two dimensional billiard system.

We reported the observation of directional emission from deformed microspheres across a surprisingly large range of deformations. For strongly deformed microspheres the far-field emission pattern is highly asymmetric and consistent with the predicted directions for refractive escape in a 2D ray tracing model. In less deformed microspheres the far-field emission pattern becomes nearly symmetric, which cannot be explained by refractive escape.

We attribute this symmetric emission pattern, which had not been previously predicted, to rapid tunneling escape of rays with an angle of incidence close to the critical angle for total internal reflection. This strong tunneling emission is directed by the non-perturbative phase space structure surrounding the stabilized four bounce island. The qualitative difference between the refractive and tunneling emission patterns is due to the presence of a stable island at the location in phase space at which escape would otherwise occur. As such, this effect is unique to resonators with index of refraction near $n \approx 1.4$.

The dynamics of the mixed phase space, consisting of intermingled regular and chaotic trajectories, were probed by measuring resonance spectra and emission patterns for a weakly deformed microsphere with different input conditions. Far-field emission patterns were found to be unchanged over a wide variety of input conditions as predicted by the ray models. This confirmed that both the asymmetric and symmetric emission patterns were the result of ray escape occurring near $\sin \chi \approx \sin \chi_c$. The Q-factors of WG modes in a deformed sphere were shown to be sensitively dependant on the initial angle of incidence indicating that chaotic diffusion is slowest near $\sin \chi \approx 1$. The shape of invariant curves near the stabilized four bounce islands was shown to lead to the excitation of qualitatively different modes depending on where light was launched in the microsphere, even for the same initial $\sin \chi$.

The optical properties of weakly deformed microspheres have been demonstrated to be heavily influenced by both ray and wave effects. The qualitative difference between refractive and tunneling emission patterns, allows the emission pattern to be

used as a sensitive probe of the interaction between wave and ray effects in the cavity.

The phase space of these weakly deformed resonators is dominated by non-chaotic trajectories. This would typically indicate that perturbation theory can be used to determine the optical properties of the resonator. We have demonstrated, however, that the influence of non-perturbative effects resulting from the broken symmetry can persist down to very small deformations. The far-field emission pattern of a quadrupolar resonator, for instance, is qualitatively different than that expected for an elliptical resonator despite the fact that the shapes differ only to second order in ϵ .

The fully three dimensional nature of deformed microspheres has also been shown to be significant. A qualitative difference exists between the classical dynamics of systems with two versus three degrees of freedom. Whereas chaotic trajectories in systems with two degrees of freedom are rigidly confined by KAM curves, this confinement is not complete for systems with more than two degrees of freedom. In this case Arnold diffusion allows any chaotic trajectory to eventually explore every finite region of the phase space. This qualitative difference has been borne out in experiments where rotationally symmetric deformed microspheres do not display the directional emission and Q-spoiling observed in nonaxisymmetric microspheres.

Future Work

The role of Arnold diffusion in a quantum or wave system is not well understood. Although it seems clear that the fully 3D nature of our deformed microspheres is an essential factor which allows the large observed changes in χ , we cannot state conclusively that this represents experimental observation of Arnold diffusion. In fact it is not clear at all how classical Arnold diffusion translates into a wave system. Arnold diffusion arises because all chaotic regions of phase space in a non-integrable system with more than two degrees of freedom are connected by the Arnold web. The filaments that make up this web of chaos can be infinitesimally small and the Arnold diffusion rate can be extremely slow, particularly in regions of phase space that contain mostly nonchaotic trajectories. The trajectory displayed in figure 35 represents a single initial condition that lead to refractive escape for the given resonator. Based on many trials it appears that most trajectories with similar $\sin \chi_0$ do not achieve refractive escape in less than 10^6 reflections. The lifetime of the trajectory in figure 35 ($\epsilon = 0.06$) would correspond to $Q \approx 10^7$ which is much greater than $Q \approx 10^4$ measured for the most deformed microsphere ($\epsilon = 0.067$). The discrepancy is even greater considering that the experimental WG mode was launched with $\sin \chi_0 \approx 1$ in the region of phase space where Arnold diffusion is slowest. The question remains open, however, as to how the quantitative results of Arnold diffusion may be modified by the wave properties of a system. The discrepancy between calculated classical trajectories and experimental

observations indicates that the lifetime of WG modes in deformed microspheres is strongly influenced by wave effects throughout the diffusion from $\sin \chi_0 \approx 1$ to $\sin \chi_0 \approx 0.7$ in addition to tunneling escape near the critical angle.

It is well documented that actual diffusion rates are often *slower* than what is calculated by purely classical methods. Classically one can consider all possible trajectories that take a system between two points in phase space and calculate an average time for the net motion to occur. When wave effects are considered, however, it is discovered that constructive interference on closed paths which return the system to its initial condition lead to an increased probability that the system state will not change. This phenomenon is referred to as dynamical localization, and effectively reduces the diffusion rate.

Typically when one calculates diffusion rates including wave effects the stationary phase approximation is employed to eliminate non-classical paths which are expected to make negligible contributions. In the context of Arnold diffusion it is often the case that the classical paths connecting two relatively nearby points in phase space are extremely long leading to slow diffusion rates along one coordinate. In the case where intrinsic material losses make extremely long trajectories physically unrealizable, it is possible that very short non-classical paths (tunneling) may dominate the diffusion rate. Furthermore, when a certain symmetry of a resonator is broken a large, new sets of non-classical trajectories would be allowed, which violate the associated conservation law. In the specific case of a deformed microsphere resonator the diffusion rate of interest is that which acts to reduce the value of $\sin \chi$. As was shown in figure 35 extremely long

classical trajectories are required to traverse a relatively short difference in $\sin \chi$. It is possible that the tunneling rate between points with slightly different $\sin \chi$ could be significantly greater than the diffusion rate predicted by the classical trajectory. In an axisymmetric resonator a tunneling event would be required to conserve the azimuthal angular momentum. Therefore, in order for a trajectory to reduce its $\sin \chi$ it must simultaneously change its inclination to the z axis by a corresponding amount in order to maintain constant L_z . This conservation law effectively requires that the trajectories before and after the tunneling event occupy somewhat different physical regions of the resonator. For the nonaxisymmetric resonator on the other hand L_z need not be conserved. This enables a trajectory to move to smaller $\sin \chi$ independently of any other changes. In particular the inclination of the trajectory can remain the same allowing more physical overlap between the original and final trajectories. Such a qualitative change, which occurs upon breaking rotational symmetry, is necessary in order to explain the experimental data.

One fundamental difference between the ray tracing picture and the real wave system is that unlike the point particle assumed in geometrical optics, a wave packet has a certain minimum extent in phase space determined by the uncertainty principle. The role of Plank's constant in determining the minimum uncertainty is played by the inverse of the size parameter. For our experimental system with large size parameter the extent of the minimum uncertainty wave packet is very small, justifying our use of geometrical optics. Despite the small extent of the wave packet, however, it is still significantly larger than much of the fine structure of the Arnold web. The finite extent of the wave packet in phase

space would effectively allow WG modes to ignore the fine structure of the Arnold web. Another way of viewing this effect is that the wave packet can sample all of the infinite number of possible initial conditions for a ray starting within its finite region of phase space. The wave packet is thus guaranteed to find any particularly direct classical paths to other regions of phase space which might be statistically impossible to find by trial and error of initial conditions.

Arnold diffusion is an intrinsically nonlinear phenomenon which only applies to chaotic trajectories. Given the fact that the optical wave equation is completely linear one might ask whether it is even meaningful to discuss the role of Arnold diffusion in an optical resonator. This question basically comes down to asking how classical the resonator is. As the size parameter increases and the system moves more towards the classical limit the minimum uncertainty wave packet shrinks and resolves more fine structure of the Arnold web. The observed fact that the diffusion rate at for large size parameters ($x \approx 800$) is much faster than that predicted by pure Arnold diffusion suggests that the diffusion rate could be further reduced in larger deformed microspheres. Using larger deformed microspheres could be a practical method of obtaining directional WG mode emission with even higher Q factors than those reported here. Conversely, one would expect the diffusion rate to increase for smaller size parameter. This would suggest that for very small microspheres only slight deformations would allow WG modes with high Q-factors to persist.

These observations highlight an interesting issue that continually surfaces when studying asymmetric resonant cavities. The boundary between geometrical and wave

optics is never clear and can depend on a number of things beyond simply comparing the wavelength to resonator size. First of all, we have demonstrated that, regardless of size parameter, one must always account for tunneling when the angle of incidence approaches the critical angle for total internal reflection. Furthermore, we have shown that even when emission is dominated by tunneling escape the directionality can be strongly influenced by the classical dynamics in the Poincaré SOS. Finally, we now see that a new important size scale is introduced by comparing the extent of the minimum uncertainty wave packet to characteristic length scales of the Arnold web. In this case the size of the wave packet is inversely related to the size parameter but the relevant length scales of the Arnold web depend sensitively on the degree of deformation and location in the SOS. In general we suggest that wave effects are more important in less deformed resonators where the Arnold web is more tightly constrained by dense sets of KAM tori. Likewise, wave effects should be more important to the diffusion of trajectories within the resonator at large $\sin \chi$. As $\sin \chi$ is reduced the thickness of the separatrix regions increases and the extent of the wave packet becomes less important.

There is a lot of theoretical work that must be done to address the issue of Arnold diffusion in a wave system. It is hoped that more understanding can be achieved, from the experimental side, by extending the experiments presented in this dissertation to smaller size parameters. At smaller sizes the importance of wave effects increases in all aspects of the resonator dynamics. Tunneling escape becomes significant even further above the critical angle as shown in figure 41 which compares the tunneling limited Q-factors in 100 μm and 50 μm radius spheres. In addition the

extent of the wave packet in phase space is inversely proportional to the size parameter. This will allow a wave packet to ignore more of the fine structure of the Arnold web, possibly leading to faster diffusion through phase space. Using the techniques described in this dissertation we have routinely fabricated nondeformed microspheres with radii less than 10 μm . Therefore, it is reasonable to expect that deformed microspheres could be produced with sizes an order of magnitude smaller than those studied here. It could be equally interesting to study much larger deformed microspheres as well.

In addition to studying the effect of size on the dynamics of deformed microspheres, there are a number of other directions for this research to be continued. Many of the effects that we have reported depend on the particular index of refraction for fused-silica. It would be very interesting to investigate the emission patterns of deformed microspheres fabricated from materials with different refractive indices. It is possible that other non-trivial directional tunneling emission patterns could be observed. A challenge that must be confronted for this research to be done is learning how to fabricate deformed microspheres from materials that do not possess the convenient physical properties of fused-silica.

A research direction that does not require new materials or fabrication techniques is a study of polarization effects in deformed microspheres. In the experiments presented here the laser was linearly polarized along the direction of the stem and the detection was polarization insensitive. In a nonaxisymmetric resonator the

eigenmodes will not, in general, be able to be classified as TE or TM modes as in the sphere. Changes in polarization of the light inside the microsphere may reveal clues as to the diffusion dynamics that lead to a ray's escape.

In conclusion we mention the need to develop deformed fused-silica microspheres for some of the applications which inspired their creation. If the previously mentioned material issues can be address, fabrication of a directional microlaser should be a straightforward process. There is no reason why deformed fused-silica microspheres cannot be used immediately to increase signal collection in cavity QED experiments. Other applications, which capitalize on the ability to use free-space coupling should be explored. With this advantage deformed microspheres may prove useful as photonics filters or chemical and biological sensors.

APPENDIX

THREE DIMENSIONAL RAY TRACING PROGRAM

The ray tracing program used to calculate Poincaré SOS plots for nonaxisymmetric 3D resonator shapes is presented. The programming language used is Fortran.

```

PROGRAM PS3D2
!Calculates surface of section data for 3D convex cavity

IMPLICIT NONE
REAL, DIMENSION(3,3) :: A, Au, Av, At
REAL, DIMENSION(3) :: Vn, Vd, Vo, V1, Vdo, Tem, Vold
REAL, DIMENSION(3) :: UP /0,0,1/, DN /0,0,-1/
REAL Chi, u, v, ug, vg, tg, ugo, vgo, tgo, cp, e, d, q, ce, s, gamma,
sinX, dm, m
REAL phi, theta, beta, alpha, Temp, Pi, Pole, DISTANCE, AXIS, Lz, px,
py, pz
REAL :: Res=10.**(-15) !, Pi=3.141592653589793
INTEGER I, J, K, L, NPTS, NITS, NIC, FLAG, dim, cnt1, cnt2, CA, Esc,
trips

OPEN (UNIT = 10, FILE = 'filename.dat')
!OPEN (UNIT = 20, FILE = 'filename2.dat')
!OPEN (UNIT = 30, FILE = ' filename3.dat')
!OPEN (UNIT = 40, FILE = ' filename4.dat')

m = 3.
dm = .4
dim = 3
AXIS = 1
NPTS = 1000
NITS = 100
NIC = 1
FLAG = 1
L = 1
CA = 1
Esc = 0
Res = 10.**(-15)
Pole = 100.*Res
cp = 2.*ASIN(1.)
Pi = cp

```

```

!SHAPE
e = 0.06           !deformation along z-axis
ce = 2.           !order of additional term
!s = -e/10        !optional sextupole term
d = 0.00          !Dipole deformation in x-y plane
q = 0.000         !Quadrupole deformation in x-y plane

Do K = 1,NIC

u = Pi/2.
v = Pi/2.         !(u,v) is starting position

Vo = R(u,v)       !Vo contains starting x,y,z coords (becomes 'present'
location)

!!!!!!Single IC

Vd(1) = 0.1        x,y,z components of initial direction vector
Vd(2) = -1.
Vd(3) = 2.3495

!!!!!!CONSTANT Lz      launches set of trajectories with same Lz

!px = -.089439      !chosen to match Lz of 6-bounce inclined .1(Lz = .084)
                    !6-bounce corresponds to py=.43825, pz=.89439

!IF (K <= 21) THEN
!  pz = SQRT(.99-px**2.) - (K+35)*1./160.*SQRT(1.-px**2.)
!ELSE
!  pz = .75*SQRT(.99-px**2.)-(K-21)*1./40*1.*SQRT(1.-px**2.)
!END IF
!py = -SQRT(1-(px**2. + pz**2.))

!Vd(1) = px
!Vd(2) = py
!Vd(3) = pz

Vd = NORM(Vd)

Vn = NORM(CROSS(Rdv(u,v),Rdu(u,v)))      !normal at starting position
Chi = ACOS(DOT_PRODUCT(Vn,-Vd))          !initial angle of incidence

IF (AXIS == 1) THEN
  CALL COORDS(Vo,phi,theta)

```

```

ELSE
  CALL COORDS2(Vo,phi,theta)
END IF

Lz = Vo(2)*Vd(3) - Vo(3)*Vd(2)

gamma = ASIN(Vd(1))

IF (dim == 3) THEN
  WRITE(10, '(f25.16," ",f20.16," ",f20.16)') phi, theta, SIN(Chi)

  ! WRITE(10, '(f25.16," ",f20.16," ",f20.16," ",f20.16)')
  phi,theta,SIN(Chi),Lz
  ! WRITE(10, '(f25.16," ",f20.16," ",f20.16," ",f20.16)')
  theta,SIN(Chi),Vo(1),gamma
  ! WRITE(10, '(f25.16," ",f20.16," ",f20.16," ",f20.16," ",f20.16)')
  phi,theta,SIN(Chi),beta,alpha
  ! WRITE(30, '(f25.16," ",f20.16," ",f20.16)') Vo(1),Vo(2),Vo(3)
  ! WRITE(30, '(f25.16," ",f20.16," ",f20.16," ",f20.16)')
  phi,theta,beta,alpha
ELSE
  !produces 2D data if trajectory confined
  to plane
  IF (phi .gt. Pi) THEN
    theta = 2.*Pi-theta
  end if
  WRITE(10, '(f25.16," ",f20.16)') theta,SIN(Chi)
  ! WRITE(20, '(f25.16," ",f20.16)') Vo(1),Vo(2)
END IF

Do I = 1,NPTS

Vd = NORM(Vd)

!Guess next reflection assuming a spherical cavity of radius |Vo|

tg = 2.*SQRT(DOT_PRODUCT(Vo,Vo))*COS(Chi) !guess distance to next point

V1 = Vo + tg*Vd !guess vector from origin to next point
V1 = V1*(1-e)
V1(3) = V1(3)/(1-e)*(1+e)
V1 = NORM(V1)

Temp = SQRT(DOT_PRODUCT(V1,V1))
vg = ACOS(V1(3)/Temp)

```

```

IF ((DIST(V1,UP) < Pole) .or. (DIST(V1,DN) < Pole)) THEN
    ug = u                                !Don't let rounding errors at pole change
    ug
ELSE IF (ABS(V1(1)) <= Res) THEN
    IF (V1(2) > 0) THEN
        ug = Pi/2.
    ELSE
        ug = 3.*Pi/2.
    END IF
ELSE
    IF ((V1(1) > Res) .and. (V1(2) >= 0)) THEN
        ug = ATAN(V1(2)/V1(1))
    ELSE IF ((V1(1) > Res) .and. (V1(2) < 0)) THEN
        ug = 2.*Pi + ATAN(V1(2)/V1(1))
    ELSE
        ug = Pi + ATAN(V1(2)/V1(1))
    END IF
END IF

DO WHILE (ug > 2.*Pi)
    ug = ug - 2.*Pi
END DO

Do WHILE (ug <= 0)
    ug = ug + 2.*Pi
END DO

tgo = tg
ugo = ug          !original guesses
vgo = vg

cnt1 = 0
cnt2 = 0

DO J = 1,NITS
    cnt1 = cnt1 + 1
    IF (cnt1 == 30) THEN      !If not converging change guess
        Print*, 'not converging'
        !READ*
        IF (cnt2 == 0) THEN
            ug = ugo + Pi
        ELSE
            ug = ugo + cnt2*.1
        END IF
        cnt1 = 0
        cnt2 = cnt2 + 1
    END IF

    IF (Any(ABS(F(ug,vg,tg,Vo,Vd)) > 10.)) THEN
        ug = ugo+.1
    
```

```

        vg = vgo
        tg = tgo
    END IF

    Tem = ABS(F(ug,vg,tg,Vo,Vd))

    IF ((Tem(1)<Res) .and. (Tem(2)<Res) .and. (Tem(3)<Res)) THEN

        cnt1 = 0
        DISTANCE = DIST(V1,Vo)
        IF (DIST(V1,Vo) < tgo/100.) THEN      !trouble shooting
            PRINT*, 'too close', FLAG        !don't find the point you
            PRINT*, ugo,vgo,tgo              !started from
            PRINT*, ug,vg,tg
            READ*
            FLAG = FLAG + 1
            IF (FLAG == 1) THEN
                ug = ugo + Pi
                vg = vgo
                tg = tgo
                PRINT*, I, 'did 1'
                READ*

                ELSE IF (FLAG == 2) THEN
                    ug = ugo
                    vg = vgo + .2*Vd(3)/ABS(Vd(3))
                    tg = tgo
                    PRINT*, 'did 2'
                    READ*

                    ELSE IF (FLAG == 3) THEN
                        ug = ugo+.2
                        vg = vgo
                        tg = tgo
                        PRINT*, 'did 3'
                        READ*

                        ELSE
                            PRINT*,I,FLAG,'Repeated Point Press any key to continue.'
                            READ*
                            FLAG = 0
                            EXIT
                        END IF
                    ELSE
                        FLAG = 0
                        EXIT
                    END IF
                END IF
            END IF
        END IF
    END IF

```

! Cramer's Rule for solving non-homogeneous liner system

```

      IF (vg < Pole) vg = Pole
      A(:,1) = Rdu(ug,vg)
      A(:,2) = Rdv(ug,vg)
      A(:,3) = -Vd
      Au = A
      Au(:,1) = -F(ug,vg,tg,Vo,Vd)
      Av = A
      Av(:,2) = -F(ug,vg,tg,Vo,Vd)
      At = A
      At(:,3) = -F(ug,vg,tg,Vo,Vd)

      ug = ug + D3(Au)/Temp
      vg = vg + D3(Av)/Temp
      tg = tg + D3(At)/Temp
      !
      !   Modify guesses for root
      !

      IF (vg < 0) THEN
        vg = -vg
        ug = ug + Pi
      END IF
      IF (vg > Pi) THEN
        vg = 2.*Pi - vg
        ug = ug + Pi
      END IF
      IF (ug > 6.*Pi) THEN
        ug = ug - AINT(ug/2./Pi)*2.*Pi
      END IF
      IF (ug < -6.*Pi) THEN
        ug = ug + AINT(-ug/2./Pi)*2.*Pi
      END IF
      DO WHILE (ug > 2.*Pi)
        ug = ug - 2.*Pi
      END DO
      DO WHILE (ug < 0)
        ug = ug + 2.*Pi
      END DO

      V1 = R(ug,vg)
    END DO

    u = ug
    v = vg
    Vold = Vo
    Vo = R(u,v)      !prep for next itteration

    IF ((Vold(3) >= 0.) .AND. (Vo(3) < 0.)) THEN
      trips = trips + 1
    END IF

    IF (AXIS == 1) THEN
      CALL COORDS(Vo,phi,theta)
    ELSE

```

```

      CALL COORDS2(Vo,phi,theta)
END IF

      Vn = NORM(CROSS(Rdu(u,v),Rdv(u,v))) !Calculate normal by cross of two
      tangents
      Vdo = Vd

      CALL REFLECTION(Vd,Vn,Chi) !Vd becomes new reflected direction
                                !note: Chi is sensitive to sign of Vn

      IF ((DIST(Vo,UP) < Pole) .or. (DIST(Vo,DN) < Pole)) THEN
          Vd = Vdo !if it hits near a pole ignore
      'reflection' and
          Vd(3) = -Vdo(3) !just act like it hit off a ceiling or
      floor
      END IF

      CALL COORDS(Vd,beta,alpha)

      Lz = Vo(2)*Vd(3) - Vo(3)*Vd(2)

      gamma = ASIN(Vd(1))

      IF (dim == 3) THEN
          WRITE(10, '(f25.16," ",f20.16," ",f20.16)') phi, theta, SIN(Chi)
!       WRITE(10, '(f25.16," ",f20.16," ",f20.16," ",f20.16)')
!           phi,theta,SIN(Chi),Lz
!       WRITE(10, '(f25.16," ",f20.16," ",f20.16," ",f20.16)')
!           theta,SIN(Chi),Vo(1),gamma
!       WRITE(10, '(f25.16," ",f20.16," ",f20.16," ",f20.16," ",f20.16)')
!           phi,theta,SIN(Chi),beta,alpha
!       WRITE(30, '(f25.16," ",f20.16," ",f20.16)') Vo(1),Vo(2),Vo(3)
!       WRITE(30, '(f25.16," ",f20.16," ",f20.16," ",f20.16)')
!           phi,theta,beta,alpha
!       WRITE(10, '(I4," ", f25.16," ", f20.16," ",f20.16)')
!           I,u,v,Chi !SIN(Chi)
      ELSE
          IF (phi .gt. Pi) THEN
              theta = 2.*Pi-theta
          end if
          WRITE(10, '(f25.16," ",f20.16)') theta,SIN(Chi)
!       WRITE(20, '(f25.16," ",f20.16)') Vo(1),Vo(2)
      END IF

      !IF (L == 100) THEN !Option to record every 100th reflection
!       WRITE(20, '(f25.16," ",f20.16," ",f20.16," ",f20.16," ",f20.16)')
!           phi,theta,SIN(Chi),beta,alpha
!       WRITE(10, '(f25.16," ",f20.16," ",f20.16)') phi,theta,SIN(Chi)

```

```

!      L = 0
!      PRINT*, I
!END IF
!L = L+1

IF ((SIN(Chi) < .689655) .AND. (CA == 1)) THEN !check for refractive
escape
    Esc = I
    CA = 2
END IF

END DO

END DO

PRINT*, ''

IF (CA == 2 ) THEN
    PRINT*, 'Escape ', Esc
ELSE
    PRINT*, 'No Escape'
END IF

PRINT*, ''
PRINT*, trips, 'round trips'
PRINT*, ''
PRINT*, 'DONE'
READ*

CLOSE (10)
CLOSE (20)
CLOSE (30)
CLOSE (40)

CONTAINS

!!!!!! Selection of various shapes

FUNCTION R(u,v)
!Calcualtes (x,y,z) coordinates of point labeled by (u,v)
    REAL, DIMENSION(3) :: R
    REAL u,v

!    ELLIPSOID
!    R(1) = (1.-e)*(1.+q)/(1.-q)*COS(u)*SIN(v)
!    R(2) = (1.-e)*SIN(u)*SIN(v)
!    R(3) = (1.+e)*COS(v)

!    NOCKEL/STONE DROP SHAPE

```

```

!   R(1)=(1.+d*(COS(v))**2+3./2.*d*(COS(v))**4)*COS(u)*SIN(v)/(1.+5/2*d)
!   R(2)=(1.+d*(COS(v))**2+3./2.*d*(COS(v))**4)*SIN(u)*SIN(v)/(1.+5/2*d)
!   R(3)=(1.+d*(COS(v))**2+3./2.*d*(COS(v))**4)*COS(v)/(1.+5/2*d)

! Quadropole & DIPOLE
!   R(1) = (1.+d*COS(cd*u)*SIN(v))*(1+e*COS(ce*v))*COS(u)*SIN(v)
!   R(2) = (1.+d*COS(cd*u)*SIN(v))*(1+e*COS(ce*v))*SIN(u)*SIN(v)
!   R(3) = (1.+e*COS(ce*v))*COS(v)

! Quad and Dipole (Optics Letters paper)
!   R(1)=(1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))*(1+e*COS(ce*v))
!           *COS(u)*SIN(v)/(1.-q)
!   R(2)=(1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))*(1+e*COS(ce*v))
!           *SIN(u)*SIN(v)/(1.-q)
!   R(3) = (1.+e*COS(ce*v))*COS(v)

! Quad and Dipole (Optics Letters paper) plus sextupole
!   R(1) = (1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))*(1+e*COS(ce*v)+s*COS(4*v))
!           *COS(u)*SIN(v)/(1.-q)
!   R(2) = (1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))*(1+e*COS(ce*v)+s*COS(4*v))
!           *SIN(u)*SIN(v)/(1.-q)
!   R(3) = (1.+e*COS(ce*v))*COS(v)

!   DOUBLE QUAD
!   R(1) = (1.+e*COS(2.*u)+d*COS(2.*v))*COS(u)*SIN(v)
!   R(2) = (1.+e*COS(2.*u)+d*COS(2.*v))*SIN(u)*SIN(v)
!   R(3) = (1.+d*COS(2.*v))*COS(v)

END FUNCTION R

FUNCTION Rdu(u,v)
!Calculates dx/du (and for Y,Z) at a point labeled by (u,v)
  REAL, DIMENSION(3) :: Rdu
  REAL u,v

!   ELLIPSOID
!   Rdu(1) = -(1.-e)*(1.+q)/(1.-q)*SIN(u)*SIN(v)
!   Rdu(2) = (1.-e)*COS(u)*SIN(v)
!   Rdu(3) = 0.

!   NOCKEL/STONE DROP SHAPE
!   Rdu(1) = -(1.+d*(COS(v))**2+3./2.*d*(COS(v))**4)
!           *SIN(u)*SIN(v)/(1.+5/2*d)
!   Rdu(2) = (1.+d*(COS(v))**2+3./2.*d*(COS(v))**4)
!           *COS(u)*SIN(v)/(1.+5/2*d)
!   Rdu(3) = 0.

!   Quad and Dipole
!   Rdu(1) = -(1.+d*COS(cd*u)*SIN(v))*(1+e*COS(ce*v))*SIN(u)*SIN(v)
!   Rdu(1) = Rdu(1) -cd*d*SIN(cd*u)*SIN(v)*(1+e*COS(ce*v))*COS(u)*SIN(v)

```

```

!   Rdu(2) = (1.+d*COS(cd*u)*SIN(v))*(1+e*COS(ce*v))*COS(u)*SIN(v)
!   Rdu(2) = Rdu(2) -cd*d*SIN(cd*u)*SIN(v)*(1+e*COS(ce*v))*SIN(u)*SIN(v)
!   Rdu(3) = 0.

!   Quad and Dipole   OL paper
!   Rdu(1) = -(1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))*(1+e*COS(ce*v))
!             *SIN(u)*SIN(v)
!   Rdu(1) = Rdu(1) -d*SIN(u)*SIN(v)*(1.+q*COS(2*u))*(1+e*COS(ce*v))
!             *COS(u)*SIN(v)
!   Rdu(1) = Rdu(1) -2*q*SIN(2*u)*(1.+d*COS(u)*SIN(v))*(1+e*COS(ce*v))
!             *COS(u)*SIN(v)
!   Rdu(1) = Rdu(1)/(1.-q)
!   Rdu(2) = (1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))*(1+e*COS(ce*v))
!             *COS(u)*SIN(v)
!   Rdu(2) = Rdu(2) -d*SIN(u)*SIN(v)*(1.+q*COS(2*u))*(1+e*COS(ce*v))
!             *SIN(u)*SIN(v)
!   Rdu(2) = Rdu(2) -2*q*SIN(2*u)*(1.+d*COS(u)*SIN(v))*(1+e*COS(ce*v))
!             *SIN(u)*SIN(v)
!   Rdu(2) = Rdu(2)/(1.-q)
!   Rdu(3) = 0.

!   Quad and Dipole           plus sextupole
!   Rdu(1) = -(1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))
!             *(1+e*COS(ce*v)+oct*COS(4*v))*SIN(u)*SIN(v)
!   Rdu(1) = Rdu(1) -d*SIN(u)*SIN(v)*(1.+q*COS(2*u))
!             *(1+e*COS(ce*v)+oct*COS(4*v))*COS(u)*SIN(v)
!   Rdu(1) = Rdu(1) -2*q*SIN(2*u)*(1.+d*COS(u)*SIN(v))
!             *(1+e*COS(ce*v)+oct*COS(4*v))*COS(u)*SIN(v)
!   Rdu(1) = Rdu(1)/(1.-q)
!   Rdu(2) = (1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))
!             *(1+e*COS(ce*v)+oct*COS(4*v))*COS(u)*SIN(v)
!   Rdu(2) = Rdu(2) -d*SIN(u)*SIN(v)*(1.+q*COS(2*u))
!             *(1+e*COS(ce*v)+oct*COS(4*v))*SIN(u)*SIN(v)
!   Rdu(2) = Rdu(2) -2*q*SIN(2*u)*(1.+d*COS(u)*SIN(v))
!             *(1+e*COS(ce*v)+oct*COS(4*v))*SIN(u)*SIN(v)
!   Rdu(2) = Rdu(2)/(1.-q)
!   Rdu(3) = 0.

!   DOUBLE QUAD
!   Rdu(1) = -(1.+e*COS(2.*u)+d*COS(2.*v))*SIN(u)*SIN(v)
!   Rdu(1) = Rdu(1) -2.*e*SIN(2.*u)*COS(u)*SIN(v)
!   Rdu(2) = (1.+e*COS(2.*u)+d*COS(2.*v))*COS(u)*SIN(v)
!   Rdu(2) = Rdu(2) -2.*e*SIN(2.*u)*SIN(u)*SIN(v)
!   Rdu(3) = 0.

```

END FUNCTION Rdu

FUNCTION Rdv(u,v)

!Calcualtes dx/dv (and for Y,Z) at a point labeled by (u,v)
 REAL, DIMENSION(3) :: Rdv

```

REAL u,v

! ELLIPSOID
! Rdv(1) = (1.-e)*(1.+q)/(1.-q)*COS(u)*COS(v)
! Rdv(2) = (1.-e)*SIN(u)*COS(v)
! Rdv(3) = -(1.+e)*SIN(v)

! NOCKEL/STONE DROP SHAPE
! Rdv(1) = (1.+d*(COS(v))**2+3./2.*d*(COS(v))**4)*COS(u)*COS(v)
! Rdv(1) = Rdv(1)-(2.*d*(COS(v))+6.*d*(COS(v))**3)*COS(u)*(SIN(v))**2
! Rdv(2) = (1.+d*(COS(v))**2+3./2.*d*(COS(v))**4)*SIN(u)*COS(v)
! Rdv(2) = Rdv(2)-(2.*d*(COS(v))+6.*d*(COS(v))**3)*SIN(u)*(SIN(v))**2
! Rdv(3) = -(1.+d*(COS(v))**2+3./2.*d*(COS(v))**4)*SIN(v)
! Rdv(3) = Rdv(3)-(2.*d*(COS(v))+6.*d*(COS(v))**3)*SIN(v)*COS(v)
! Rdv = Rdv/(1.+5/2*d)

! Quad and Dipole
! Rdv(1) = (1.+d*(COS(cd*u)*SIN(v))*(1+e*(COS(ce*v))*COS(u)*COS(v)
! Rdv(1) = Rdv(1) +d*(COS(cd*u)*COS(v)*(1+e*(COS(ce*v))*COS(u)*SIN(v)
! Rdv(1) = Rdv(1) -ce*e*SIN(ce*v)*(1.+d*(COS(cd*u)*SIN(v))
!           *COS(u)*SIN(v)
! Rdv(2) = (1.+d*(COS(cd*u)*SIN(v))*(1+e*(COS(ce*v))*SIN(u)*COS(v)
! Rdv(2) = Rdv(2) +d*(COS(cd*u)*COS(v)*(1+e*(COS(ce*v))*SIN(u)*SIN(v)
! Rdv(2) = Rdv(2) -ce*e*SIN(ce*v)*(1.+d*(COS(cd*u)*SIN(v))
!           *SIN(u)*SIN(v)
! Rdv(3) = -(1.+e*(COS(ce*v))*SIN(v)
! Rdv(3) = Rdv(3) -ce*e*SIN(ce*v)*COS(v)

! Quad and Dipole OL paper
! Rdv(1) = (1.+d*(COS(u)*SIN(v))*(1.+q*(COS(2*u)))*(1+e*(COS(ce*v))
!           *COS(u)*COS(v)
! Rdv(1) = Rdv(1) +d*(COS(u)*COS(v)*(1.+q*(COS(2*u)))*(1+e*(COS(ce*v))
!           *COS(u)*SIN(v)
! Rdv(1) = Rdv(1) -ce*e*SIN(ce*v)*(1.+d*(COS(u)*SIN(v))*(1.+q*(COS(2*u))
!           *COS(u)*SIN(v)
! Rdv(1) = Rdv(1)/(1.-q)
! Rdv(2) = (1.+d*(COS(u)*SIN(v))*(1.+q*(COS(2*u)))*(1+e*(COS(ce*v))
!           *SIN(u)*COS(v)
! Rdv(2) = Rdv(2) +d*(COS(u)*COS(v)*(1.+q*(COS(2*u)))*(1+e*(COS(ce*v))
!           *SIN(u)*SIN(v)
! Rdv(2) = Rdv(2) -ce*e*SIN(ce*v)*(1.+d*(COS(u)*SIN(v))*(1.+q*(COS(2*u))
!           *SIN(u)*SIN(v)
! Rdv(2) = Rdv(2)/(1.-q)
! Rdv(3) = -(1.+e*(COS(ce*v))*SIN(v)
! Rdv(3) = Rdv(3) -ce*e*SIN(ce*v)*COS(v)

! Quad and Dipole plus saextupole
Rdv(1) = (1.+d*(COS(u)*SIN(v))*(1.+q*(COS(2*u))
!           *(1+e*(COS(ce*v))+oct*(COS(4*v))*COS(u)*COS(v)
Rdv(1) = Rdv(1) +d*(COS(u)*COS(v)*(1.+q*(COS(2*u))
!           *(1+e*(COS(ce*v))+oct*(COS(4*v))*COS(u)*SIN(v)
Rdv(1) = Rdv(1) -ce*e*SIN(ce*v)*(1.+d*(COS(u)*SIN(v))

```

```

      *(1.+q*COS(2*u))*COS(u)*SIN(v)
Rdv(1) = Rdv(1) -4*oct*SIN(4*v)*(1.+d*COS(u)*SIN(v))
      *(1.+q*COS(2*u))*COS(u)*SIN(v)
Rdv(1) = Rdv(1)/(1.-q)
Rdv(2) = (1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))
      *(1.+e*COS(ce*v)+oct*COS(4*v))*SIN(u)*COS(v)
Rdv(2) = Rdv(2) +d*COS(u)*COS(v)*(1.+q*COS(2*u))
      *(1.+e*COS(ce*v)+oct*COS(4*v))*SIN(u)*SIN(v)
Rdv(2) = Rdv(2) -ce*e*SIN(ce*v)*(1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))
      *SIN(u)*SIN(v)
Rdv(2) = Rdv(2) -4*oct*SIN(4*v)*(1.+d*COS(u)*SIN(v))*(1.+q*COS(2*u))
      *SIN(u)*SIN(v)
Rdv(2) = Rdv(2)/(1.-q)
Rdv(3) = -(1.+e*COS(ce*v)+oct*COS(4*v))*SIN(v)
Rdv(3) = Rdv(3) -ce*e*SIN(ce*v)*COS(v)
Rdv(3) = Rdv(3) -4*oct*SIN(4*v)*COS(v)

!   DOUBLE QUAD
!   Rdv(1) = (1.+e*COS(2.*u)+d*COS(2.*v))*COS(u)*COS(v)
!   Rdv(1) = Rdv(1) -2.*d*SIN(2.*v)*COS(u)*SIN(v)
!   Rdv(2) = (1.+e*COS(2.*u)+d*COS(2.*v))*SIN(u)*COS(v)
!   Rdv(2) = Rdv(2) -2.*d*SIN(2.*v)*SIN(u)*SIN(v)
!   Rdv(3) = -(1.+d*COS(2.*v))*SIN(v)-2.*d*SIN(2.*v)*COS(v)

```

END FUNCTION Rdv

```

FUNCTION F(u,v,t,Vo,Vd)
!F is the vector function to be solved to find the next reflection
  REAL, DIMENSION(3) :: Vo, Vd, F
  REAL u,v,t
  F = R(u,v)- Vo - Vd*t
END FUNCTION F

```

```

SUBROUTINE REFLECTION(Vd,Vn,Chi)
!Given incident vector, Vd, surface normal, Vn
!Calculates reflected vector, Vd, and angle of incidence, Chi
  REAL, DIMENSION(3), INTENT(INOUT) :: Vd, Vn
  REAL, INTENT(OUT) :: Chi
  Vd = NORM(Vd)
  Vn = NORM(Vn)
  Chi = Pi-ACOS(DOT_PRODUCT(Vd,Vn))
  Vd = Vd - 2*(DOT_PRODUCT(Vd,Vn))*Vn
END SUBROUTINE REFLECTION

```

```

SUBROUTINE COORDS(V,phi,theta)
!Given the vector V, calculate spherical polar coordinates
!theta is measured from +z axis
!phi is measured from +x axis towards +y axis
  REAL, DIMENSION(3), INTENT(IN) :: V

```

```

REAL, INTENT(OUT) :: phi,theta
REAL mag
mag = SQRT(DOT_PRODUCT(V,V))
theta = ACOS(V(3)/mag)
IF (ABS(V(1)) <= Res) THEN
  IF (V(2) > 0) THEN
    phi = Pi/2.
  ELSE
    phi = 3.*Pi/2.
  END IF
ELSE
  IF ((V(1) > Res) .and. (V(2) >= 0)) THEN
    phi = ATAN(V(2)/V(1))
  ELSE IF ((V(1) > Res) .and. (V(2) < 0)) THEN
    phi = 2.*Pi + ATAN(V(2)/V(1))

  ELSE
    phi = Pi + ATAN(V(2)/V(1))
  END IF
END IF
END SUBROUTINE COORDS

SUBROUTINE COORDS2(V,phi,theta)
!Given the position parameters u,v calculate spherical polar
  coordinates
!theta is measured from +x axis
!phi is measured from -z axis towards +y axis
REAL, DIMENSION(3), INTENT(IN) :: V
REAL, INTENT(OUT) :: phi,theta
REAL mag
mag = SQRT(DOT_PRODUCT(V,V))
theta = ACOS(V(1)/mag)
IF (ABS(V(2)) <= Res) THEN
  IF (V(3) > 0) THEN
    phi = Pi/2.
  ELSE
    phi = 3.*Pi/2.
  END IF
ELSE
  IF ((V(2) > Res) .and. (V(3) >= 0)) THEN
    phi = ATAN(V(3)/V(2))
  ELSE IF ((V(2) > Res) .and. (V(3) < 0)) THEN
    phi = 2.*Pi + ATAN(V(3)/V(2))
  ELSE
    phi = Pi + ATAN(V(3)/V(2))
  END IF
END IF
IF (phi < 3.*Pi/2.) THEN
  phi = phi + Pi/2.
ELSE
  phi = phi - 3.*Pi/2.
END IF

```

```

END SUBROUTINE COORDS2

FUNCTION NORM(V3)
!Normalizes a 3D vector
  REAL, DIMENSION(3) :: V3, NORM
  NORM = V3 / SQRT(DOT_PRODUCT(V3,V3))
END FUNCTION NORM

FUNCTION DIST(V1,V2)
!Calculates the distance between points at the ends of two vectors
  REAL, DIMENSION(3) :: V1, V2, DIFF
  REAL DIST
  DIFF = V1 - V2
  DIST = SQRT((DIFF(1))**2 + (DIFF(2))**2 + (DIFF(3))**2)
END FUNCTION DIST

FUNCTION CROSS(V1, V2)
!Calculates the cross product (V1 x V2) of two 3D vectors
  REAL, DIMENSION(3) :: V1, V2, CROSS
  REAL, DIMENSION(2,3) :: B
  B(1,:) = V1
  B(2,:) = V2
  CROSS(1) = D2(B(:,2:3))
  CROSS(2) = -(D2(B(:,1:3:2)))
  CROSS(3) = D2(B(:,1:2))
END FUNCTION CROSS

FUNCTION D2(N)
!Calculates the determinant of a 2x2 matrix
  REAL D2
  REAL, DIMENSION(2,2) :: N
  D2 = N(1,1)*N(2,2) - N(1,2)*N(2,1)
END FUNCTION D2

FUNCTION D3(M)
!Calculates the determinant of a 3x3 matrix by cofactor expansion
  REAL D3
  REAL, DIMENSION(3,3) :: M
  D3 = M(1,1)*D2(M(2:3,2:3)) - M(1,2)*D2(M(2:3,1:3:2)) +
M(1,3)*D2(M(2:3,1:2))
END FUNCTION D3

END PROGRAM PS3D2

```

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