SEGAL'S OPEN COVERING THEOREM

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The following result is [S, Proposition A.5]:

Theorem 1.1. *Let $\mathcal{C}$ be a small category and let X be a space. Let $\mathrm{Op}(X)$ denote the poset of open subsets of X, regarded as a category in the usual way. Let $\Phi\colon \mathcal{C} \to \mathrm{Op}(X)$ be a functor, and consider the composite map $\mathrm{hocolim}_{\mathcal{C}}\, \Phi \to \mathrm{colim}_{\mathcal{C}}\, \Phi \to X$. For each $x \in X$ let $\mathcal{C}_x$ be the full subcategory consisting of those c for which $x \in \Phi(\mathcal{C})$. If each $|\mathcal{C}_x|$ is contractible, then $\mathrm{hocolim}_{\mathcal{C}}\, \Phi \to X$ is a weak homotopy equivalence.*

Unfortunately, the key step from the proof in [S] has some nontrivial typos that obscure the beauty of Segal's argument (see Remark 1.5 below). The point of this short note is to clarify those issues, while at the same time giving a slightly expanded treatment of the details.

For ease of presentation we restrict to the case where $\mathcal{C}$ is a poset P. This has the advantage that $|\mathcal{C}|$ is a simplicial complex. One can get around this issue (as in [S]) by replacing all geometric realizations with fat realizations.

For a map $p\colon E \to B$ and $b \in B$, let $E_b = p^{-1}(b)$. Segal defines a map p to be **almost locally trivial** (a.l.t. for short) if for every $b \in B$ there is an open neighborhood of E_b in E that is homeomorphic (as spaces over B) to an open neighborhood of $E_b \times \{b\}$ in $E_b \times B$. The pullback of an a.l.t. map is also a.l.t. The key to Segal's proof is the following:

Proposition 1.2. *An a.l.t. map with contractible fibers is a weak homotopy equivalence.*

Proof that Proposition 1.2 implies Theorem 1.1. For us Δ^n will denote the simplex $\{(t_0,\ldots,t_n) \mid \sum_i t_i = 1\}$. We equip each of these with the metric induced from $\mathbb{R}^n$. These particular models of simplices have the technical advantage that the face inclusions preserve the metric.

Let P be a finite poset and let $\Phi\colon P \to \mathrm{Op}(Z)$ be a functor satisyfing the hypotheses of Theorem 1.1. The simplicial complex $|P|$ is obtained by gluing together standard simplices, and so it makes sense to talk about distance between two points in a common simplex. Fix $0 < \epsilon < \frac{1}{2}$ (the exact value does not matter), and for every simplex σ in $|P|$ let U_σ be the collection of all points of $|P|$ whose distance to a point of σ is less than ϵ. Then U_σ is an open neighborhood of σ in $|P|$ that deformation-retracts down to σ.

Recall that an n-simplex in $|P|$ is a chain $\sigma = [\sigma_0 < \sigma_1 < \cdots < \sigma_n]$ in P. Then

$$\operatorname*{hocolim}_{P} \Phi = \bigcup_{n,\sigma} \Delta^n_\sigma \times \Phi(\sigma_0) \subseteq |P| \times Z.$$

Segal considers the diagram

$$\begin{tikzcd} \operatorname{hocolim}_P \Phi \arrow[r, equal] & \bigcup_{n,\sigma} \Delta^n \times \Phi(\sigma_0) \arrow[r, tail, "\subseteq"] \arrow[dr] & \bigcup_{n,\sigma} U_\sigma \times \Phi(\sigma_0) \arrow[r, tail, "\subseteq"] \arrow[d, "\pi"'] & {|P| \times X} \arrow[dl, "\pi_2"] \\ & & Z. & \end{tikzcd}$$

The first inclusion is a weak equivalence by inspection, and the map π is a.l.t. with contractible fibers. So Proposition 1.2 implies that π is a weak equivalence, hence so is $\operatorname{hocolim}_P \Phi \to Z$. $\square$

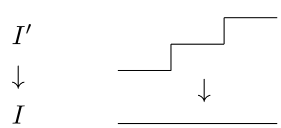
The rest of this note is aimed at helping readers understand the proof of Proposition 1.2. A.l.t. maps with contractible fibers need not be Serre fibrations, but they almost are. Suppose $E \to B$ is a.l.t. with contractible fibers, and consider a lifting diagram

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, tail] & E \arrow[d, "p"] \\ I \arrow[r, "f"] & B. \end{tikzcd}$$

Using the a.l.t. property at $0 \in I$ we can produce a lifting λ defined on some interval $[0, t]$. Using the a.l.t. property at $t \in I$ we can produce a lifting λ_1 of f defined on some interval $[t', t'']$ containing t in the interior, but one need not have $\lambda(t') = \lambda_1(t')$ and so we cannot patch the lifts together. However, $\lambda(t')$ and $\lambda_1(t')$ lie in the same fiber, and by hypothesis that fiber is contractible. So there is a path joining them, and we can use this to connect λ to λ_1. Proceeding in this way, we can produce a diagram

$$\begin{tikzcd} \{0\} \arrow[r, equal] \arrow[d, tail] & \{0\} \arrow[r] \arrow[d, tail] & E \arrow[d, "p"] \\ I' \arrow[r] \arrow[urr, dotted, "\lambda"] & I \arrow[r] & B \end{tikzcd}$$

where I' is a 'zig-zag interval' projecting onto I as follows:



The lift λ is defined on the horizontal portions using the a.l.t. property, and on the vertical portions using the contractibility of the fibers. One should think of I' as a kind of "blowup" of I, where certain points have been blown up into vertical line segments; we have proven that the lifting property works modulo passing to a blowup. Note that this immediately implies that an a.l.t. map with contractible fibers induces an isomorphism on π_0 and a surjection on π_1. To extend this result to higher π_k, one needs an analogous property that replaces I with higher-dimensional cubes. In fact, Segal shows the following:

Lemma 1.3. *Let $E \to B$ be a.l.t. with contractible fibers. Then for every finite polyhedral complex X and subcomplex A, and every diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & E \arrow[d, "p"] \\ X \arrow[r] & B \end{tikzcd}$$

there exists a weak homotopy equivalence $(X', A') \to (X, A)$ and a commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, tail] & A \arrow[r] & E \arrow[d, "p"] \\ X' \arrow[r] \arrow[urr, "\lambda"] & X \arrow[r] & B. \end{tikzcd}$$

Proof that Lemma 1.3 implies Proposition 1.2. Let $e \in E$ be a chosen basepoint and let $b = p(e)$. Suppose $f \in \pi_k(B, b)$ and choose a representative $f\colon S^k \to B$. Consider the diagram

$$\begin{tikzcd} \{e\}' \arrow[r] \arrow[d] & \{e\} \arrow[r] & f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ (S^k)' \arrow[rr, "\simeq"'] & & S^k \arrow[r] & B. \end{tikzcd}$$

The map $f^*E \to S^k$ is a.l.t. with contractible fibers, so the lemma implies that one can get a lifting $(S^k)' \to f^*E$ for an appropriate choice of $(S^k)'$. But then the canonical element in $\pi_k(S^k)$ lifts to $\pi_k((S^k)')$, and hence to $\pi_k(f^*E)$. Pushing into E gives a lift of the original element f, proving that $\pi_k E \to \pi_k B$ is surjective.

The proof of injectivity is similar, starting with a diagram

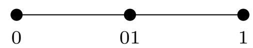
$$\begin{tikzcd} S^k \arrow[r, "g"] \arrow[d, tail] & E \arrow[d, "p"] \\ D^{k+1} \arrow[r] & B. \end{tikzcd}$$

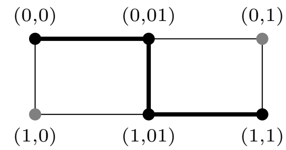
Pull back the bundle to D^{k+1}, use the lemma to produce a lifting on a weakly equivalent space $(D^{k+1})' \to D^{k+1}$, and then observe that this proves the homotopy element g is trivial. $\square$

It remains to prove Lemma 1.3, and this is the heart of the argument. Let X be a finite cell complex that is a union of closed cells each of which is homeomorphic to a disk. Let Q_X be the poset of the cells and inclusions, and let $\mathbb{D}(X)$ be the nerve of Q_X. Note that the 0-simplices of $\mathbb{D}(X)$ are the cells of X. For a cell σ in X let $D\sigma \subseteq \mathbb{D}(X)$ be the union of all simplices whose vertices are cells that contain σ. Finally, define $X' \subseteq X \times \mathbb{D}(X)$ to be the union of the subspaces $\sigma \times D\sigma$ where σ ranges over all the cells of X. Composing with the projection $X \times \mathbb{D}(X) \to X$ gives a map $X' \to X$.

Example 1.4. Suppose that $X = I$, with vertices 0 and 1 and edge 01. The following picture shows $\mathbb{D}(X)$ and $X' \subseteq X \times \mathbb{D}(X)$ (X' is shown in bold):

$\mathbb{D}(X)$



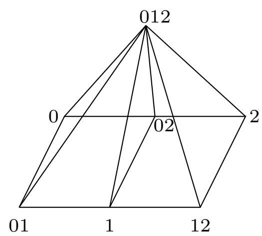


$X \times \mathbb{D}(X)$

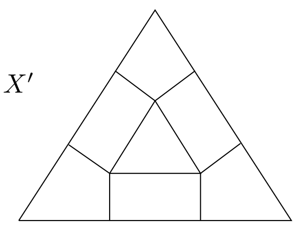
Observe that $D(0) \subseteq \mathbb{D}(X)$ is the edge joining 0 to 01, $D(1)$ is the edge joining 1 to 01, and $D(01)$ is the single vertex 01. The space X' is formed from $0 \times D(0)$, $1 \times D(1)$ and $01 \times D(01)$ (the latter is the vertical segment).

Next suppose $X = \Delta^2$. Then $\mathbb{D}(X)$ and X' are shown below:

$\mathbb{D}(X)$



X'



Note that the base of $\mathbb{D}(X)$ is one-dimensional, and $\mathbb{D}(X)$ is the cone on that base. The space X' consists of seven two-dimensional cells, corresponding to the seven cells of Δ^2. For example, $D(012)$ is the cone-point of $\mathbb{D}(X)$ and $[012] \times D(012)$ is the triangle in the middle of X'. $D(0) \subseteq \mathbb{D}(X)$ consists of two triangles glued along an edge, and then $0 \times D(0)$ is the quadrilateral in the lower left corner of X' (the common edge of the two triangles is not drawn there). $D(01) \subseteq \mathbb{D}(X)$ is the edge joining 01 to 012, and $01 \times D(01)$ is the rectangle at the bottom of X'. The map $X' \to X$ can be identified with the map that retracts X' down to its center simplex in the evident way (e.g. the rectangles are squashed onto an edge, and the nonregular quadrilaterals are squashed to points).

Segal called X' the **explosion** of X. Observe that it is indeed a kind of blowup.

Returning to the general situation, if σ is a cell of X then $D(\sigma)$ is a cone on a space $L(\sigma)$. Precisely, $L(\sigma)$ is the subcomplex of $\mathbb{D}(X)$ consisting of simplices whose vertices properly contain σ. One can also identify $L(\sigma)$ with $\mathbb{D}(\mathrm{Lk}\,\sigma)$ where $\mathrm{Lk}\,\sigma \subseteq X$ is the usual link of σ. One can build X' by starting with $\sigma \times D\sigma$ where σ has maximal dimension and then gluing on $\tau \times D\tau$ for lower-dimensional τ via the subspace $\tau \times L\tau$. For example, when $X = \Delta^2$ one starts with the middle 2-simplex, glues on the three rectangles, and finally the three quadrilaterals; the V-shapes where each quadrilateral is glued correspond to $L(0)$, $L(1)$, and $L(2)$ in $\mathbb{D}(X)$.

Finally, we give the broad sketch of the proof of Lemma 1.3. Start by subdividing X into small pieces so that on each piece *some* lifting can be produced using the a.l.t. condition. These liftings do not agree, but they may be regarded as a partial lifting defined on the "middle cells" inside of X' (these are the cells $\sigma \times D\sigma$ where σ has maximal dimension). One extends the lifting to other cells $\tau \times D\tau$ by a descending induction on the degree of τ. At each step one uses the contracibility of the fibers together with induction to produce the desired extension. Further details are in [S].

Remark 1.5. [S] uses D for both $\mathbb{D}(X)$ and $D\sigma$, which can be disorienting. There is a typo in the description of $D\sigma$ where the vertices are claimed to be the cells contained in σ, rather than the cells containing σ. Finally, $L(\sigma)$ is mis-identified as $\mathbb{D}(\partial\sigma)$ rather than $\mathbb{D}(\operatorname{Lk}\sigma)$. These three issues can make it hard for readers to parse what X' actually is.

References

[S] G. Segal, *Classifying spaces related to foliations*, Topology **17** (1978), 367–382.