COMPREHENSIVE SCHOOL MATHEMATICS PROGRAM

ELEMENTS OF MATHEMATICS

BOOK 1

INTRODUCTORY LOGIC

CEMREL, Inc.

Chapter 1

THE FORMAL LANGUAGE

1.1 Introduction

In this book we begin a study of logic. It is our hope that by the time you complete Book 1 you will agree with us that logic is an interesting subject. It is certainly a very old subject. The Greek philosopher Aristotle (384-322 B.C.) is thought to have been the first to study the subject seriously. He was the first to state formal laws and rules of logic. Indeed, Aristotle's work on logic was an outstanding intellectual achievement that dominated Western thinking for over 2000 years.

The century and a half following Aristotle's death (300-150 B.C.) was an enormously creative period for Western logic. Interest then waned for about 1000 years. A second creative period occurred during the 11th through the 15th century when the Scholastics revived interest in logic. Again interest waned until the middle of the 19th century when a third period of creativity began, a period that has continued until the present day.

Each creative period seems to have been sparked by efforts to focus logic on some new and special problem. The Greek logicians were philosophers who were interested in a broad spectrum of problems in the physical and social sciences. The Scholastics were also philosophers who were interested in a variety of things, but they were especially interested in problems in Christian theology. Many modern logicians are mathematicians who are concerned with the logical foundations of mathematics, and much of the logic of the 19th and 20th centuries has been mathematical logic.

Since we are interested in logic for mathematical purposes we will look at logic from a mathematical viewpoint. That is, we will view logic as it is used by mathematicians with its own subject matter and its own techniques and methodology. What is the subject matter of mathematical logic? In our study, we propose to focus on those arguments that mathematicians call "proofs". Just how these "arguments" are to become objects of study will be explained more fully as we proceed. For the moment let us consider the question of why we wish to study mathematical proofs. One reason is simply to understand what proofs are and how proofs are made. Such an understanding is important for any student of mathematics because proving things is an essential part of what a professional mathematician does. It is, however, not all that he does. Let us illustrate our point with an analogy.

Imagine an explorer named "Jones" who is planning an expedition into some remote and unknown region. Jones is a member of the Explorers' Club, which is a club composed of amateur and professional explorers. The Explorers' Club encourages Jones and indeed provides him with some financial assistance. With this encouragement and support Jones prepares for his trip. There are many things he must do. He must arrange for food and shelter, provide for safety and health, and prepare as well as he can to cope with any problems that he may encounter along the way.

When his plans are finally complete Jones' expedition sets out. Let us suppose that the expedition is a very successful one. Deep in a hitherto unexplored region of a tropical jungle Jones discovers a __________ [*]. Eagerly Jones returns to report his discovery to the Explorers' Club. But how is Jones to convince those skeptics who question the truth of his claims? It may be that all Jones can do to convince some people is to challenge them to go and see for themselves. To help those who wish to do this Jones will provide a map. Of course the purpose of the map is to help people find what Jones has discovered. In order for the map to do this it must have on it some place that is well-known. Furthermore the map must show, step by step, how to proceed from the known location in order to arrive at the location where the discovery was made.

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[*] What Jones discovered is not essential to our argument. Consequently the reader may fill in the blank with anything his imagination can conjure up. It does not matter if different readers fill in the blank in different ways. Indeed it is not even necessary to fill in the blank at all, simply read "___" as "blank".

A mathematician is also a kind of explorer. He does not explore the physical world but a world of thought, a world composed of things we call mathematical objects, such as numbers, points, and lines. To prepare for a mathematical expedition a mathematician does not stockpile food, medicine, and clothing; he stockpiles information. He begins by learning what is known about the subject of interest.

Let us suppose that a mathematician named "Smith" becomes interested in geometry. Smith's first step is to go to the library and read about geometry and the vast number of things that are known about it. When he feels that he knows the literature quite well he then begins to reflect on what he knows and to think about geometry. Let us suppose that Smith is also very successful. One day he is studying a figure and he observes something interesting. He considers another example and observes the same thing. After considering a number of examples Smith reflects on his observations and convinces himself that he has discovered a new geometric property.

Smith is a member of the American Mathematical Society, a society of amateur and professional mathematicians. Consequently Smith immediately arranges to present his discovery at the Society's next meeting. How is Smith to deal with his detractors? How is he to answer those who question the truth of his claim? Clearly there is no place that Smith can take people, point and say, "See", for mathematical truths are "seen" only by the mind's eye.

When Smith appears before the Society to explain his new discovery he will present a proof. A proof is a mathematical map that shows how to proceed from known information in order to "see" the truth of a new result. The purpose of a proof is to convince people of the truth of a claim. Why mathematical arguments convince is one of the questions we hope to answer in this book. In order to do this we must explain a variety of things, such as what a mathematician means when he says that a certain statement follows from another; what he means by the claim that a given argument is logically sound; or, if an argument is not sound, what it means to say that the argument contains an error.

Since the purpose of a proof is to convince us of the truth of some claim we should perhaps begin by explaining what we mean when we say that a given statement is true. After a brief reflection we see that this is such a basic idea that we have difficulty explaining it in simpler terms. Nevertheless, we offer the following explanation. A statement is true provided what the statement asserts is in agreement with the facts; that is, provided what the statement asserts is indeed the case. How then are we to decide whether a given statement is true? We submit that in the final analysis all anyone can do is to examine the facts and form an opinion. Thus, for example, when a person asserts that

Mark Twain wrote Tom Sawyer

that person can at best be claiming that it is his judgement that this is the case. Confronted with evidence to the contrary he might change his mind. In fact, it is not difficult to give an example of a statement about which many people have changed their minds.

Not so many years ago it was commonly taught that

Columbus discovered America.

Most people interpreted this statement as meaning that Columbus was the first European to set foot on land in North America, and they accepted this statement as true. Later, however, it was learned that the Vikings were here long before Columbus was.

Should those who were students during this period be disturbed that they had been taught something that was not so? Probably not. That the Vikings were here first does not detract from Columbus' contribution. Furthermore, those who taught that Columbus discovered America were more than likely merely acting on the best information that they had at the time.

This situation does, however, pose an interesting question. If we accept the view that certainty is denied us then how should we make truth judgements about statements so that we have reasonable confidence in those judgements?

For many questions we have come to accept the so-called scientific method. Consider, for example, the statement

Cancer is caused by a virus.

We consider it proper to reserve judgement on this proposition awaiting the results of studies presently being made at the various cancer research centers throughout the world. In short, we should not make a judgement until we know what the facts are. Even such a simple proposition as

It is raining outside

should be judged only after checking the local weather conditions.

Suppose, however, we are asked to judge the claim

It is raining outside or it is not raining outside,

or the claim

It is raining outside and it is not raining outside.

We would surely agree that the first statement is true and the second is false. Furthermore we make this judgement without even a glance out of the window. Our decision is based on our understanding of English and the proper use of the words "and", "or", and "not".

Thus there is an important difference between our approach to the simple statement

It is raining outside

and our approach to the compound sentence

It is raining outside or it is not raining outside.

For the first statement it is clear that the meaning of the sentence must be taken into account in deciding its truth or falsity.

In this book truth and falsity will be denoted by "t" and "f" respectively and be referred to collectively as "truth values". The problem of determining whether a given sentence is true or false will be referred to as "the problem of determining the truth value of the sentence". As the foregoing examples illustrate, we sometimes judge the truth value of a sentence from its meaning, and we sometimes judge its value from its sentence structure. Sometimes we do both. Indeed we tend to judge the truth value of a compound sentence on the basis of the truth values of its component parts and our understanding of the connecting words.* Hence, it is important that we agree on the use of these connecting words.

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* We refrain from using the correct grammatical word "conjunction" in this context as we are reserving this word for a technical use below.

That part of logic which deals with relationships between compound sentences and their component parts is called the propositional calculus. In the propositional calculus we are not interested in how the truth values of simple sentences are determined but rather in how the truth values of simple sentences determine the truth values of compound ones. Consequently we can simplify our study by using a special language that enables us to focus our attention on the way the sentences are compounded. It is the goal of this chapter to describe for you this special language that we call the language of the propositional calculus.

We call the symbols of the propositional calculus together with the rules for their use a "language" because of its marked similarities with "natural" languages such as English. Let us explain more fully what we mean. The English language is built up from an alphabet together with some special symbols called "punctuation marks": periods, commas, etc. Letters of the alphabet may be combined to form words, but not every combination of letters is a word. English spelling is so bizarre that no one is ever likely to succeed in formulating a simple rule that will distinguish those combinations of letters which are words from those which are not. Instead we compile word-lists (called "dictionaries") and agree that a combination of letters is a word if and only if it appears on some such list.

Words and punctuation marks may be combined to form sentences; but, once again, not every such combination is a sentence. Nevertheless, at the sentence level things are easier than at the word level in that we do not need a sentence-list to tell us which combinations of words and punctuation marks are sentences. There are rules that help us distinguish sentences from non-sentences. For example, one such rule is that every sentence must have a predicate and a subject.

As we pointed out earlier, the English language contains connecting words such as "and" and "or" that can be used to compound sentences, and the propositional calculus is concerned with the study of such sentence compounds. It is not concerned with spelling; it is not concerned with sentence composition in general; but rather it deals with sentences which have been compounded in certain ways that we will discuss in a moment.

Collectively, the special symbols used in the propositional calculus are called an alphabet. The alphabet of the propositional calculus consists of letters

P, Q, R, ...

(called propositional variables), some symbols which are called connectives, and some other symbols that play the role of punctuation marks (these last two sorts of symbols are described later).

In the propositional calculus, the propositional variables "P", "Q", "R", ... play the role of basic sentences, that is, sentences that are not to be subjected to any further analysis. It is customary to refer to these propositional variables and those "sentences" which can be built up from them using the connectives and "punctuation marks" as well formed formulas (wff's) or simply formulas. Then it is natural to call the propositional variables atomic formulas. We will use this nomenclature from this point on. The succeeding sections explain how compound formulas may be built up from atomic formulas.

1.2 Negation

The English language is a very rich language that provides us with a variety of ways to express the same idea. In particular there are lots of ways to deny, or negate, a sentence. Consider, for example, the sentence

(1) Fire is hot.

If, for some strange reason, you wish to disagree with this assertion you might reply

(2) Fire is not hot

or

(3) Fire isn't hot

or

(4) Fire is never hot

or

(5) Not fire is hot.

Each of the sentences (2) - (5) is a denial (or negation) of sentence (1). Since sentence (1) is true each of the denial sentences must be false. Indeed, that is the essential idea of denial: the denial of a true sentence is a false sentence and the denial of a false sentence is a true sentence.

In the propositional calculus we will not be interested in the variety of ways that a sentence can be denied and consequently, in our special language, we provide only one way to negate a formula. The form of denial we choose is illustrated by sentence (5) above.

We adjoin to our language a special symbol "$\sim$" that we call *the negation symbol*. A formula of the propositional calculus is denied, or negated, by placing the negation symbol "$\sim$" before it. For example, "$\sim P$" is *the* negation of "P". More generally, if $\mathcal{S}$ is any formula of the propositional calculus, be it an atomic formula like "P", or a formula built up from atomic formulas in ways we have not yet explained, then $\sim\mathcal{S}$ (read "not $\mathcal{S}$") is also a formula of the propositional calculus, a formula that we call "*the* negation of $\mathcal{S}$". In the formula $\sim\mathcal{S}$, we say that "$\sim$" is the *principal connective*. The significance of this is explained in the next section.

1.3 Conjunction and Disjunction

Negation provides one way to make new formulas from old ones. Another way involves the use of the symbol "$\wedge$" which we adjoin to the language of the propositional calculus with the intent that it play the role of the English connecting word "and". From any two formulas, for example, "P" and "$\sim Q$" we can obtain a formula "$[P \wedge \sim Q]$" that we call *the conjunction of "P" and "$\sim Q$"*. The symbols "[" and "]" serve simply as punctuation marks and are to be thought of as part of the language of the propositional calculus. It should be mentioned that we also permit a formula to be compounded with itself, e.g., "$[P \wedge P]$". While this may seem like a strange and useless compound, it is one that occurs naturally in certain contexts. It is more convenient to accept it than to avoid it.

In general, if $\mathcal{S}$ and $\mathcal{T}$ are formulas of the propositional calculus, then $[\mathcal{S} \wedge \mathcal{T}]$ (read "$\mathcal{S}$ and $\mathcal{T}$") is also a formula of the propositional calculus, a formula that we call *the conjunction of $\mathcal{S}$ with $\mathcal{T}$*.

We will adopt the practice of omitting brackets from formulas when such omission will cause no confusion. For example, were we to write

(1) $P \wedge [Q \wedge R]$

instead of

(2) $[P \wedge [Q \wedge R]]$

everyone would know exactly what was intended. Indeed, since (1) is less cluttered with symbols it may in fact be more readily understood than (2). Notice that we *cannot* omit the brackets that remain in (1). Conjunction, as described above, is something one does to *two* formulas. The remaining brackets are essential to indicate that the two formulas which are being compounded by conjunction in this case are the atomic formula "P" and the nonatomic formula "$[Q \wedge R]$".

In the formula $[\mathcal{S} \wedge \mathcal{T}]$, we say that "$\wedge$" is the principal connective. We are now in a position to provide a nontrivial example of this idea, which will be of use to us a little later. The idea is a simple one. A formula that has been built up from atomic formulas may involve many connectives. The principal connective is simply that connective which is used last when one builds the formula from scratch.

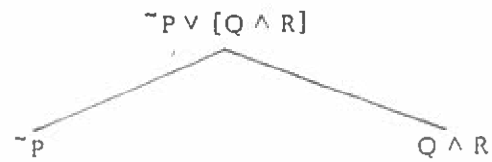
Example 1: "$\sim[P \wedge Q]$" is built up from the atomic formulas "P" and "Q" as follows: first form "$[P \wedge Q]$", then form "$\sim[P \wedge Q]$". Hence "$\sim$" is the last connective used, and is thus the principal connective.

Example 2: "$\sim P \wedge [Q \wedge R]$" is built up from the atomic formulas "P", "Q", and "R" as follows: first form "$\sim P$" and "$[Q \wedge R]$", then form "$\sim P \wedge [Q \wedge R]$". Hence the left hand "$\wedge$" is the last connective used, and is thus the principal connective.

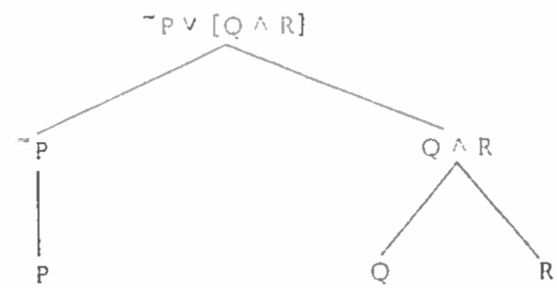
To play the role of the English connecting word "or" we introduce the special symbol "$\vee$". If $\mathcal{S}$ and $\mathcal{T}$ are formulas of the propositional calculus, then so also is $[\mathcal{S} \vee \mathcal{T}]$ (read "$\mathcal{S}$ or $\mathcal{T}$"). We call the formula $[\mathcal{S} \vee \mathcal{T}]$ *the disjunction of $\mathcal{S}$ with $\mathcal{T}$*, and in that formula, we say that "$\vee$" is the principal connective.

The idea of principal connective is basic to a method of analyzing formulas by decomposing them into their atomic parts. The tool we use for this purpose is the

<u>decomposition tree</u>, by means of which we can show exactly how the appropriate atomic formulas are assembled to produce a given formula. We illustrate with the formula "$\sim P \vee [Q \wedge R]$". We recognize that the principal connective of this formula is "$\vee$". This final stage in the construction of the given formula is pictured in the following diagram:



which shows clearly that "$\sim P \vee [Q \wedge R]$" is compounded from "$\sim P$" and "$Q \wedge R$". Next we observe that the principal connective of "$\sim P$" is "$\sim$", and the principal connective of "$Q \wedge R$" is "$\wedge$". So we extend the diagram to show this previous stage in the construction of the formula "$\sim P \vee [Q \wedge R]$".



Since "P", "Q", and "R" are atomic formulas we can decompose "$\sim P \vee [Q \wedge R]$" no further, and the decomposition tree is complete.

Exercise 1: In each of the following formulas identify the principal connective and then draw the decomposition tree for the formula:

a) $P \wedge [Q \vee R]$

b) $[P \wedge Q] \vee R$

c) $[P \vee \sim R] \vee P$

d) $\sim[P \vee [Q \wedge P]]$

e) $\sim P \vee [Q \wedge P]$

f) $[P \vee [P \wedge P]] \vee P$

1.4 Sentences and Well Formed Formulas

One reason for calling expressions like " $P \wedge Q$ " formulas rather than sentences is that we wish to think of the formula " $P \wedge Q$ " as describing a pattern for English sentences. To obtain an example of a sentence of the pattern " $P \wedge Q$ " we replace "P" by some declarative sentence,* " $\wedge$ " by "and", and "Q" by a declarative sentence. For example, if "P" is replaced by

(1) The Moon is made of Liederkranz

and "Q" is replaced by

(2) Old Man Moses is dead

we obtain

(3) The Moon is made of Liederkranz and Old Man Moses is dead.

In the future we will indicate a replacement like (1) by writing simply

(4) P: The Moon is made of Liederkranz.

We read (4) as "P, The Moon is made of Liederkranz".

We call (3) a replacement instance, or simply an instance, of the formula " $P \wedge Q$ " and we call " $P \wedge Q$ " a pattern for the sentence (3). When forming instances of formulas we will draw upon both English sentences and mathematical sentences. Here are a few instances of the formula " $P \wedge \tilde{~}Q$ ":

(5) A beagle is a dog and a parakeet is not a dog.

(6) The number 4 is prime and the number 3 is not a prime.

(7) A beagle is a dog and a beagle is not a dog.

(8) $4 + 5 < 8$ and $3 \neq 6$.

(9) Dogs bark and frogs do not fly.

Although the formula " $P \wedge \tilde{~}Q$ " involves two different propositional variables we are permitted to replace each variable by the same sentence. Thus (7) is an instance of

* A declarative sentence is one which makes a forthright statement, such as "Your eyes are like limpid pools". Declarative sentences are to be distinguished from other sorts of sentences such as interrogative sentences (which ask questions: "Are you trying to get fresh with me?") and imperative sentences (which give commands: "Never darken my doors again!").

" $P \wedge {\sim}Q$ ". It is also an instance of " $P \wedge {\sim}P$ ". However, of the sentences (5) - (9), only (7) is an instance of " $P \wedge {\sim}P$ " because we follow the convention that each occurrence of the same variable in a formula must be replaced by the same sentence. That is, if a variable, say "P", occurs in more than one place in a formula, then to produce an instance of that formula we must use the same sentence in each place where the variable "P" occurs.

As we said above, when forming instances of formulas we are permitted to mix English sentences and mathematical ones. For example, consider the formula " ${\sim}P \vee [Q \wedge R]$ " with

P: $2 + 2 = 4$

Q: George Washington was the first President of the United States

R: The rain in Spain falls mainly in the plain.

We have

(10) $2 + 2 \neq 4$; or George Washington was the first President of the United States and the rain in Spain falls mainly in the plain.

At this point you might object that our theory has run amuck. No one makes sentences like (10). The rules of good English reject such sentences; indeed we do not even have the means to punctuate such a sentence properly. But keep in mind that our long-range objective is not to study English but to study mathematical arguments. Furthermore we are viewing logic as applied to mathematics. Consequently we do not wish our study to be confined to sentence compounds that happen to be considered good English at the present time. Rather we wish the propositional calculus to encompass sentences that might be acceptable at some future time, sentences that are possible in English, German, French, or any other of the natural languages that exist now or that might exist in the future. Therefore we will study any and all "sentences" that can be compounded according to the rules we specify in the propositional calculus—even compounds that are not represented by reasonable English sentences.

Exercise 2: Write five instances of the formula " ${\sim}P \vee Q$ ".

Not only does every formula identify a sentence pattern but every declarative sentence is an instance of some formula. For example,

(11) A beagle is a dog or a beagle is not a cat

is an instance of " $P \vee {}^{\sim}Q$ " with

P: A beagle is a dog

Q: A beagle is a cat.

Of course, (11) is also an instance of " $P \vee Q$ " with

P: A beagle is a dog

Q: A beagle is not a cat.

Furthermore, (11) is also an instance of the atomic formula "P" with

P: A beagle is a dog or a beagle is not a cat.

Exercise 3: Identify a pattern for each of the following sentences so that the sentences which are to replace the propositional variables appearing in the "pattern" formula are as simple as possible.

Example: Of the three patterns given above for the sentence "A beagle is a dog or a beagle is not a cat", only " $P \vee {}^{\sim}Q$ " satisties the conditions of this exercise.

a) David slew Goliath and Goliath is dead.

b) You can't not go to school.

c) Romeo was not alive or Juliet was deceived.

d) It's not true that I didn't do no work.

1.5 Truth Tables

In Section 1.1 we mentioned our intention to refer to truth (t) and falsity (f) collectively as "truth values". We further said that the problem of determining whether a given sentence is true or false would be referred to as the problem of determining the truth value of that sentence. One objective of our study of the propositional calculus is to find methods of making truth value judgements in which we can have confidence. One approach to this problem is to use the formulas of the propositional calculus to classify sentences and then see what conclusions we can draw about the sentences in a given category. For example, consider the formula "$\sim P$". This formula is a pattern for all those sentences that are obtained by negating other sentences. If we replace "P" by a true sentence, we obtain a false sentence; if we replace "P" by a false sentence, we obtain a true sentence. This is a direct consequence of what it means to negate.

Let us try to transform this information about sentences into information about the formula "$\sim P$". For the purposes of the propositional calculus, it is of no real concern to us which specific sentences are substituted for "P"; all we really care about is whether they are true or false. So instead of talking about replacing "P" by a sentence, let us talk about assigning "P" a truth value. Assigning "P" the value t is the analogue of replacing "P" by a true sentence, and assigning "P" the value f is the analogue of replacing "P" by a false sentence. We can then summarize the essential idea of negation in the following truth table:

P	$\sim P$
t	f
f	t

Here the column headed "P" lists all possible truth values that can be assigned to "P". The order in which these values are listed is not important. The essential thing is that the atomic formula "P" can be assigned the value t or the value f and these are the only assignments possible. In the column headed "$\sim P$" are listed the truth values for "$\sim P$" that correspond to the various possible assignments of values for "P". For example, the first row of the table tells us that if "P" is assigned the value t then the value of "$\sim P$" will necessarily be f. This merely translates what we said earlier: if the "P" in "$\sim$P" is replaced by a true sentence, we will obtain a false sentence.

If the observations we have just made about negation seem so obvious and trivial that it is hard to imagine their leading to anything important, we ask you to be patient just a little longer as we explain more fully what we are about. Our general plan is as follows: If a truth value is assigned to each atomic formula of the propositional calculus, then the meanings which we intend for "$\sim$", "$\wedge$", "$\vee$" (and any other connectives we will introduce later) will determine the truth values of every other formula of the propositional calculus. So it is clear that our immediate task is to clarify the way in which these truth values of nonatomic formulas are determined, that is, to make sure that we all agree on the meanings of "$\sim$", "$\wedge$", "$\vee$", etc.

We have already reached agreement on negation. If $\mathcal{S}$ is any formula at all, then $\mathcal{S}$ and $\sim\mathcal{S}$ must have opposite truth values. We summarize this in a truth table which we will call <u>the truth table for negation</u>.

(N)

$\mathcal{S}$	$\sim\mathcal{S}$
t	f
f	t

We intend the connective "$\wedge$" to be used in the sense of the English connecting word "and". Consequently if $\mathcal{S}$ and $\mathcal{T}$ are formulas of the propositional calculus, we specify that $\mathcal{S} \wedge \mathcal{T}$ has the value t iff* each of the component formulas $\mathcal{S}$ and $\mathcal{T}$ has the value t:

(C)

$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S} \wedge \mathcal{T}$
t	t	t
t	f	f
f	t	f
f	f	f

* The phrase "if and only if" is used so frequently in mathematical discussions that we find it convenient to introduce an abbreviation for it. So from here on whenever you see "iff" you should understand it as abbreviating "if and only if".

This is the truth table for conjunction. In the first two columns we have listed all possible combinations of truth values for $\mathcal{S}$ and $\mathcal{T}$. For two formulas there are exactly four combinations of values: both formulas could be assigned the value t; the first could be assigned the value t and the second the value f; the first could be assigned the value f and the second the value t; or both formulas could be assigned the value f. The order in which these four combinations are listed is of no importance. We have made the list in a systematic way above because that helps avoid overlooking a combination. Such a safe-guard is worth having when one begins to deal with larger truth tables. Whether the column for $\mathcal{S}$ is listed first, or that for $\mathcal{T}$, is also unimportant. The second row of the table, for example, tells us that if $\mathcal{S}$ is a formula that has the value t and $\mathcal{T}$ is a formula that has the value f, then the formula $\mathcal{S} \wedge \mathcal{T}$ must have the value f. And this is so whether you think of $\mathcal{S}$ having the value t before or after you think of $\mathcal{T}$ having the value f.

To play the role of the English connecting word "or" we introduced the special symbol "$\vee$". It may be the case that we all agree on the proper use of the English connecting word "or", but to be sure that we do let us consider the following examples.

Suppose that in a discussion of early U.S. history someone whose memory is not too good says

(1) George Washington was President or George Washington was Secretary of State.

To this we would probably reply, "Yes, he was President but he was never Secretary of State." The "yes" in our reply indicates that we accept (1) as true.

Similarly we would accept

(2) Alexander Hamilton was President or Alexander Hamilton was Secretary of the Treasury

as true because it is true that Alexander Hamilton was Secretary of the Treasury, though false that he was President.

Now consider the claim that

(3) Benjamin Franklin was President or Benjamin Franklin was Vice President.

To this we would surely reply "No. He was neither President nor Vice President." Here our "no" reply indicates that we view (3) as false.

As a final example consider the assertion

(4) Thomas Jefferson was President or Thomas Jefferson was Secretary of State.

To this someone might reply, "Yes, in fact he was both" and someone else might reply "No, he was both." Clearly each respondent agrees that each component sentence in (4) is true but they are thinking about "or" in different ways.

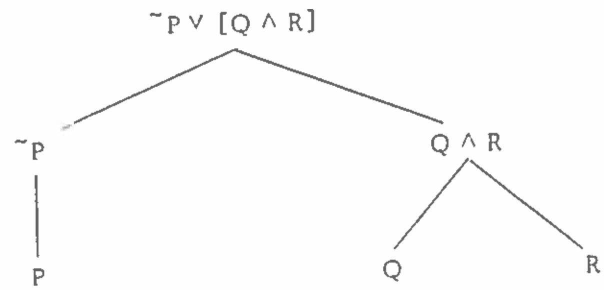
The person who answer "no" is interpreting the assertion (4) to mean that Thomas Jefferson held exactly one of the offices of President and Secretary of State but not both. This use of "or" we call the exclusive "or". The person who replied "yes" is interpreting the assertion (4) to mean that Thomas Jefferson held at least one (and hence possibly both) of these two offices. This use of "or" we call the inclusive "or".

For the propositional calculus (and indeed for mathematics in general) we find the inclusive "or" to be the most useful one and hence we specify that for formulas $\mathcal{S}$ and $\mathcal{T}$ the disjunction $\mathcal{S} \vee \mathcal{T}$ shall have the value t in all cases except when $\mathcal{S}$ and $\mathcal{T}$ each have the value f.

($\mathcal{D}$)

$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S} \vee \mathcal{T}$
t	t	t
t	f	t
f	t	t
f	f	f

We will now show how, if each atomic formula is assigned a truth value, the truth tables ($\mathcal{N}$), ($\mathcal{C}$), and ($\mathcal{D}$) determine a truth value for every formula that is compounded from atomic formulas using "$\sim$", "$\wedge$", and "$\vee$". Consider, for example, the formula "$\sim P \vee [Q \wedge R]$". We begin by forming the decomposition tree for the given formula:



This decomposition tree shows that our given formula is constructed from the atomic formulas "P", "Q", and "R" by first forming "~P" and "Q ∧ R", and then forming "~P ∨ [Q ∧ R]". We will construct a truth table for "~P ∨ [Q ∧ R]" by following the same order of construction.

We begin with the three atomic formulas "P", "Q", and "R". Since each of these three atomic formulas could be assigned either of the values t and f, there are eight (= 2 × 2 × 2) different combinations of truth values that can be assigned. We begin our table by listing these eight combinations:

P	Q	R
t	t	t
t	t	f
t	f	t
t	f	f
f	t	t
f	t	f
f	f	t
f	f	f

Once again, we point out that the order in which these columns appear is not important, but it is usually helpful to put them in the same order in which the variables appear in the formula under consideration, if that is possible.

We now move up to the next level of the decomposition tree and add a column for "~P". The values in this column are determined from the column headed "P" using table (η).

P	Q	R	~P
t	t	t	f
t	t	f	f
t	f	t	f
t	f	f	f
f	t	t	t
f	t	f	t
f	f	t	t
f	f	f	t

Similarly we add a column for " Q ∧ R ", determining these values from the columns headed "Q" and "R" using table (C).

P	Q	R	~P	Q ∧ R
t	t	t	f	t
t	t	f	f	f
t	f	t	f	f
t	f	f	f	f
f	t	t	t	t
f	t	f	t	f
f	f	t	t	f
f	f	f	t	f

Once again, it makes no difference whether we put the " ~P " column before or after the " Q ∧ R " column, but, as before, it is convenient to list the component formulas " ~P " and " Q ∧ R " in the same order in which they occur in the formula under consideration.

Finally, we move up to the highest level of the decomposition tree and add a column for " ~P ∨ [Q ∧ R] ", determining these values from the " ~P " and " Q ∧ R " columns using table (B).

P	Q	R	$\sim P$	$Q \wedge R$	$\sim P \vee [Q \wedge R]$
t	t	t	f	t	t
t	t	f	f	f	f
t	f	t	f	f	f
t	f	f	f	f	f
f	t	t	t	t	t
f	t	f	t	f	t
f	f	t	t	f	t
f	f	f	t	f	t

This is the truth table for "$\sim P \vee [Q \wedge R]$". And what does this table tell us? It tells us many things. For example, suppose that someone walks up to you on the street and says, "If in the formula '$\sim P \vee [Q \wedge R]$' I replace 'P' by a true sentence and I replace 'Q' and 'R' by false sentences, will the resulting sentence be true?" In order to respond, you have only to look at the above truth table, note that in row 4 "P" is assigned the value t and "Q" and "R" are each assigned the value f with the result that "$\sim P \vee [Q \wedge R]$" has value f. So you reply, "No; as a matter of fact the sentence you will produce will be false." Your inquisitor might then continue, "Gee, will that always happen? Will I always get a false sentence no matter if I replace 'P', 'Q', and 'R' by true or false sentences?" To answer this question you have only to scan the last column of the truth table. You will see five "t's" and three "f's" and so you can respond, "No, you won't always get a false sentence. In fact if you make substitutions using various combinations of true and false sentences and keep a record of the truth values of the resulting sentences, you will find that true sentences occur slightly more frequently than false ones."

PROBLEM SET 1

In each of Problems 1 - 20 make a truth table for the given formula.

1. $\sim[\sim P]$
2. $P \wedge \sim Q$
3. $\sim[P \wedge Q]$
4. $\sim P \wedge \sim Q$
5. $\sim[\sim[\sim P]]$
6. $\sim[\sim P \wedge \sim Q]$
7. $\sim P \wedge Q$
8. $P \vee \sim Q$
9. $\sim P \vee \sim Q$
10. $\sim[\sim P \vee \sim Q]$
11. $\sim P \vee Q$
12. $[P \wedge Q] \wedge R$
13. $P \wedge [Q \wedge R]$
14. $[P \vee Q] \vee R$
15. $P \vee [Q \vee R]$
16. $P \vee [Q \wedge R]$
17. $P \wedge [Q \vee R]$
18. $[P \vee Q] \wedge [P \vee R]$
19. $[P \wedge Q] \vee [P \wedge R]$
20. $[\sim P \vee Q] \wedge R$

21. Choose your own symbol to denote the exclusive "or" and make its truth table.

22. Each of the following sentences is an instance of a well formed formula. Determine the truth value of the sentence by referring to the truth table of the corresponding formula.

    a) The Potomac River flows through Little Rock, Arkansas, or the Potomac River does not flow through Little Rock, Arkansas.

    b) Pigs have wings and 2 + 2 = 4.

    c) Hawaii is a state of the Union and [Chicago is north of New Orleans or the population of New York City is 123].

    d) [Hawaii is a state of the Union or Chicago is north of New Orleans] and the population of New York City is 123.

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1.6 Implication

In Section 1.3 we introduced the connectives "$\wedge$" and "$\vee$". Our purpose in including them in the language of the propositional calculus is to enable us to use the propositional calculus to study "and/or" sentence compounds. Another way of learning something about such compounds is to study the following operational system.

Let "τ" denote the set of truth values, i.e., $\tau = \{t, f\}$. Corresponding to the two connectives "$\wedge$" and "$\vee$" we define two binary operations on τ:

$\wedge$	t	f
t	t	f
f	f	f

$\vee$	t	f
t	t	t
f	t	f

Anything we can discover about the operational system (τ, $\vee$, $\wedge$) will tell us something about "$\wedge/\vee$" formulas of the propositional calculus and hence something about "and/or" sentence compounds in general. For example, it is easily seen from the operation tables above that $\wedge$ and $\vee$ are each commutative. This tells us that, given any formulas $\mathcal{S}$ and $\mathcal{T}$ of the propositional calculus, the formulas $\mathcal{S} \wedge \mathcal{T}$ and $\mathcal{T} \wedge \mathcal{S}$ will have the same truth values, regardless of the values assigned to their atomic parts. We could have established this by considering the following truth table, whose last two columns are identical:

$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S} \wedge \mathcal{T}$	$\mathcal{T} \wedge \mathcal{S}$
t	t	t	t
t	f	f	f
f	t	f	f
f	f	f	f

The operations $\wedge$ and $\vee$ are but two of the sixteen different operations that could be defined on τ. Why sixteen? Consider the following blank table:

*	t	f
t		
f		

To define an operation ★ on $\mathcal{T}$ we have only to write either "t" or "f" in each of the four empty boxes. Since we have two choices for each of the four boxes, we can fill all four boxes in 2^4 different ways. Hence there are sixteen different operations that can be defined on $\mathcal{T}$. Of course most of these sixteen different binary operations on the set of truth values do not correspond to any connecting word in any known language. For example consider

	t	f
t	t	t
f	t	t

	t	f
t	f	f
f	f	f

The first table would correspond to a connecting word that produces a true sentence from every pair of sentences, and the second would correspond to a connecting word that produces a false sentence from every pair of sentences. Certainly, no such words exist in English, or indeed in any other known language.

The fact that there are no such connecting words in any known language would not prevent us from inventing appropriate connectives and adjoining them to the language of the propositional calculus. Indeed, in a later section we look at an interesting connective that has no English counterpart. For the moment, however, we will study one particular operation on $\mathcal{T}$ that does have a reasonable English interpretation. Consider the following operation table:

$\Rightarrow$	t	f
t	t	f
f	t	t

As a correspondent to this operation on $\mathcal{T}$ we adjoin to the language of the propositional calculus a connective "$\Rightarrow$" that we call the <u>conditional</u> (or <u>implication</u>). If $\mathcal{S}$ and $\mathcal{T}$ are formulas of the propositional calculus, then $\mathcal{S} \Rightarrow \mathcal{T}$ is also a formula of the propositional calculus. We read "$\mathcal{S} \Rightarrow \mathcal{T}$" as "$\mathcal{S}$ implies $\mathcal{T}$". We call $\mathcal{S}$ the <u>antecedent</u> and $\mathcal{T}$ the <u>consequent</u> of the conditional $\mathcal{S} \Rightarrow \mathcal{T}$, and in the formula $\mathcal{S} \Rightarrow \mathcal{T}$ we say that "$\Rightarrow$" is the principal connective. The truth table for the conditional is as follows:

(3)

$\mathcal{S}$	$\mathcal{J}$	$\mathcal{S} \Rightarrow \mathcal{J}$
t	t	t
t	f	f
f	t	t
f	f	t

We are interested in the conditional because it has been found to be useful in analyzing certain kinds of mathematical arguments. It also corresponds in part to the English use of " If ... then ... " in sentences of the form

If John is older than Mary, then John is older than Bill.

Let us illustrate.

Suppose that the Apple Bobbing Club is electing a president. The candidates are Jones and Smith and Smith is expected to win. The vote is taken and when the last ballot is dropped in the box everyone turns to the club sponsor and asks, "Is Smith the president?" The sponsor, shocked that his students know so little about democratic procedures, answers, perhaps with a tone of irritation in his voice,

(1) If Smith has a majority of the votes, then Smith is the president.

So the votes are counted, Smith has a majority, and he is given the oath of office.

Under these circumstances, we would surely agree that the sponsor has correctly informed his students, which is to say that (1) is true. Suppose, however, that after counting the votes and finding that Smith has a majority someone recalls that Smith is president of Gum Chewing and a school rule forbids anyone from being president of two clubs. Consequently Jones, who was runner-up, is made president. Under these circumstances the students would surely be justified in saying to their advisor, "You were wrong. You should have said,

(2) If Smith has a majority of the votes and Smith is not president of another club, then Smith is the president."

Suppose on the other hand that when the vote is counted it is found that Smith does not have a majority and Jones is given the oath of office, or Smith does not have a majority but Jones is disqualified and Smith is made president. Are either of these situations in conflict with (1)? We claim not. We hold that (1) is true under all circumstances but one— Smith has a majority but is not made president.

From the operation table for $\Rightarrow$ we see that it is not a commutative operation. That is, "$P \Rightarrow Q$" and "$Q \Rightarrow P$" need not have the same truth value. But that is clear from the following instances of these two formulas:

If Smith lives in Chicago, then Smith lives in Illinois

If Smith lives in Illinois, then Smith lives in Chicago

the first of which is true and the second false. At this point we introduce some new terminology: If $\mathcal{S}$ and $\mathcal{T}$ are well formed formulas, then $\mathcal{T} \Rightarrow \mathcal{S}$ is called the *converse* of $\mathcal{S} \Rightarrow \mathcal{T}$ It is not uncommon for people to confuse a conditional with its converse. But, as we have seen, a conditional and its converse need not have the same truth value. Consequently, making such a confusion can lead to logical mistakes, and if such a mistake is made, then it is often said that one has made *the error of the converse*.

PROBLEM SET 2

In Problems 1 - 20 make a truth table for the given formula.

1. $P \Rightarrow {}^{\sim}Q$
2. $P \Rightarrow P$
3. ${}^{\sim}Q \Rightarrow {}^{\sim}P$
4. $[Q \wedge P] \Rightarrow Q$
5. $Q \Rightarrow [Q \wedge P]$
6. $Q \wedge [P \Rightarrow Q]$
7. $[Q \vee S] \Rightarrow Q$
8. $Q \Rightarrow [Q \vee P]$
9. $Q \vee [P \Rightarrow Q]$
10. $Q \vee {}^{\sim}Q$
11. $Q \wedge {}^{\sim}Q$
12. $[P \vee Q] \Rightarrow [{}^{\sim}Q \vee P]$
13. $[P \Rightarrow Q] \Rightarrow [[P \vee R] \Rightarrow Q]$
14. $[P \Rightarrow Q] \Rightarrow [[P \wedge R] \Rightarrow Q]$
15. $[{}^{\sim}P \wedge [P \vee Q]] \Rightarrow Q$
16. $[P \wedge [Q \vee {}^{\sim}Q]] \Rightarrow P$
17. ${}^{\sim}P \Rightarrow [P \Rightarrow Q]$
18. ${}^{\sim}P \Rightarrow [P \wedge Q]$
19. $[P \Rightarrow Q] \vee [Q \Rightarrow P]$
20. $P \Rightarrow [P \Rightarrow Q]$

21. Suppose that we know that " $P \Rightarrow Q$ " is true and we know that "Q" is true. What can we say about the truth or falsity of "P" ?

22. Each of the following sentences is an instance of a well formed formula. Determine the truth value of the sentence by referring to the truth table of the corresponding formula.

    a) If Kansas City is west of St. Louis then 1975 was a leap year.

    b) $2 + 2 \geq 0$ does not imply that hummingbird eggs weigh three tons.

    c) $3 \cdot_7 4 = 3$ or $[3 \cdot_7 4 = 6$ implies that $3 \cdot_7 4 = 4]$

    d) If Canada is in South America and the first person to walk on the Moon was a Russian, then a whale is a mammal.

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1.7 Tautologies; Equivalence of Formulas

Before we go farther into the study of the propositional calculus we should perhaps pause to explain where we are going and what we hope to accomplish. We explained at the outset that one purpose of the propositional calculus is to enable us to study those mathematical arguments known as "proofs". By an argument we simply mean any statement or collection of statements that is intended to convince us of something. To study such an argument we will look at the patterns of the sentences involved. If these patterns are well formed formulas of the propositional calculus, then by studying the formulas of the propositional calculus we might be able to learn something about this argument in particular, and arguments in general. One thing we will observe in our study is that arguments also have patterns. That is, arguments are made by fitting sentences together in certain patterns. But before we can say much more about this we must discuss a few more ideas.

Recall our earlier observation that the operation $\wedge$ is commutative. This fact assures us that, whatever truth values are assigned to "P" and "Q", the resulting truth value of "$P \wedge Q$" will be the same as that of "$Q \wedge P$". That is, if "$P \wedge Q$" has truth value t then "$Q \wedge P$" has truth value t, and if "$Q \wedge P$" has truth value t, then "$P \wedge Q$" has truth value t. Similarly, if "$P \wedge Q$" has truth value f then "$Q \wedge P$" has truth value f, and if "$Q \wedge P$" has truth value f, then "$P \wedge Q$" has truth value f. In light of this discussion it would be interesting to look carefully at the truth table of the following formula:

(1) $$[[P \wedge Q] \Rightarrow [Q \wedge P]] \wedge [[Q \wedge P] \Rightarrow [P \wedge Q]].$$

Let us examine the truth table of formula (1).

P	Q	$P \wedge Q$	$Q \wedge P$	$[P \wedge Q] \Rightarrow [Q \wedge P]$	$[Q \wedge P] \Rightarrow [P \wedge Q]$	$[[P \wedge Q] \Rightarrow [Q \wedge P]] \wedge [[Q \wedge P] \Rightarrow [P \wedge Q]]$
t	t	t	t	t	t	t
t	f	f	f	t	t	t
f	t	f	f	t	t	t
f	f	f	f	t	t	t

Notice that the formulas "$[P \wedge Q] \Rightarrow [Q \wedge P]$", "$[Q \wedge P] \Rightarrow [P \wedge Q]$", and "$[[P \wedge Q] \Rightarrow [Q \wedge P]] \wedge [[Q \wedge P] \Rightarrow [P \wedge Q]]$" each have truth value t no matter what truth values are assigned to the variables "P" and "Q". There are many other formulas that have this property. We call such formulas <u>tautologies</u>. That is, a tautology is a formula that has truth value t regardless of the truth values assigned to its component atomic formulas. As may be seen from this example, we can use truth tables to determine in a purely mechanical way whether a given formula is a tautology. For example, each of the following is a tautology:

a) $P \vee \tilde{}P$

b) $[P \wedge Q] \Rightarrow P$

c) $\tilde{}[P \wedge \tilde{}P]$

d) $[\tilde{}P \wedge [P \vee Q]] \Rightarrow Q$

To show this we need only form their truth tables. We provide one table and leave the others as exercises for the reader.

P	Q	$P \wedge Q$	$[P \wedge Q] \Rightarrow P$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	t

Exercise 4: By forming their truth tables, show that formulas (a), (c), and (d) above are tautologies.

We are interested in tautologies for a variety of reasons. Since one of the objectives of logic is to enable us to separate true sentences from false ones we might jump to the conclusion that we are interested in tautologies because they are formulas that identify patterns of sentences each of which is true. But a little reflection on the nature of a tautology makes it clear that while they do identify true sentences those sentences do not contain much information. Let us illustrate.

A certain Jones is a zoologist who specializes in the study of monkeys. One day we rush up to him with our new technique for identifying true sentences and explain:

(2) Monkeys have fleas or monkeys don't have fleas

and we know this is true because it is an instance of the tautology

$$P \vee {}^{\sim}P.$$

Needless to say Jones will not be impressed. He will surely agree that (2) is true, even if he has never heard of a tautology, but will object that it is of no interest because it does not tell him anything about monkeys.

No, we are interested in tautologies for other reasons. For one thing, tautologies help us study argument patterns by telling us about certain relationships between formulas. One such relationship is that of <u>equivalence</u>. Two formulas $\mathcal{S}$ and $\mathcal{T}$ are <u>equivalent</u> if and only if the truth value of $\mathcal{S}$ is the same as that of $\mathcal{T}$ for all possible assignments of truth values to the propositional variables involved. For example, from the first four columns in the truth table on page 27, we see that the formulas "$P \wedge Q$" and "$Q \wedge P$" are equivalent since for all possible assignments of truth values to "P" and "Q", "$P \wedge Q$" and "$Q \wedge P$" have the same truth value.

It seems reasonable to ask if there is a formula involving two formulas $\mathcal{S}$ and $\mathcal{T}$ which is true if and only if both $\mathcal{S}$ and $\mathcal{T}$ are true or both $\mathcal{S}$ and $\mathcal{T}$ are false. That is, a formula based on the following truth table:

(*)

$\mathcal{S}$	$\mathcal{T}$	
t	t	t
t	f	f
f	t	f
f	f	t

One such formula is $[\mathcal{S} \Rightarrow \mathcal{T}] \wedge [\mathcal{T} \Rightarrow \mathcal{S}]$; that is, the conjunction of a conditional with its converse. The reader should check that this formula has the above truth table.

Now suppose $\mathcal{S}$ and $\mathcal{T}$ are "$P \wedge Q$" and "$Q \wedge P$" respectively. Then the formula based on truth table (*) is "$[[P \wedge Q] \Rightarrow [Q \wedge P]] \wedge [[Q \wedge P] \Rightarrow [P \wedge Q]]$". Since we have already determined that "$P \wedge Q$" and "$Q \wedge P$" always have the same truth values no matter what truth values are assigned to "P" and "Q" (i.e., they are equivalent formulas), we can be certain that only the first and fourth lines of truth table (*) are applicable to the formula

$$[[P \wedge Q] \Rightarrow [Q \wedge P]] \wedge [[Q \wedge P] \Rightarrow [P \wedge Q]].$$

In other words, this formula is a tautology.

The general situation regarding equivalent formulas is the same as in this special example. That is, two formulas $\mathcal{S}$ and $\mathcal{T}$ are equivalent iff

(3) $$[\mathcal{S} \Rightarrow \mathcal{T}] \wedge [\mathcal{T} \Rightarrow \mathcal{S}]$$

is a tautology.

Since we must deal with formulas like (3) quite frequently in the material ahead we find it convenient to introduce a special symbol, "$\Leftrightarrow$", called the biconditional, that can be used to simplify both the notation in (3) as well as the task of producing truth tables for formulas like (3).

We will use the formula

(4) $$\mathcal{S} \Leftrightarrow \mathcal{T},$$

read "$\mathcal{S}$ if and only if $\mathcal{T}$", as an abbreviation for (3). Formula (4) is not a well formed formula of our language but is a shorthand way of writing (3). In other words, whenever you see a formula like (4) it should make you think of the well formed formula (3). Notice that the truth table for the biconditional is (*) above which we repeat here for emphasis.

($\mathcal{B}$)

$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S} \Leftrightarrow \mathcal{T}$
t	t	t
t	f	f
f	t	f
f	f	t

If we use the biconditional, then our earlier truth table on page 27 (which, as we have already mentioned, established the equivalence of "$P \wedge Q$" and "$Q \wedge P$") can be simplified:

P	Q	$P \wedge Q$	$Q \wedge P$	$[P \wedge Q] \Leftrightarrow [Q \wedge P]$
t	t	t	t	t
t	f	f	f	t
f	t	f	f	t
f	f	f	f	t

We can now make use of the biconditional to help us characterize equivalent formulas: Two formulas $\mathcal{S}$ and $\mathcal{T}$ are equivalent iff their biconditional $\mathcal{S} \Leftrightarrow \mathcal{T}$ is a tautology.

Example: " $\tilde{}[P \wedge Q]$ " and " $P \Rightarrow \tilde{}Q$ " are equivalent.

In order to establish this we must show that

$$\tilde{}[P \wedge Q] \Leftrightarrow [P \Rightarrow \tilde{}Q]$$

is a tautology. This is easily shown by a truth table:

P	Q	$\tilde{}Q$	$P \wedge Q$	$\tilde{}[P \wedge Q]$	$P \Rightarrow \tilde{}Q$	$\tilde{}[P \wedge Q] \Leftrightarrow [P \Rightarrow \tilde{}Q]$
t	t	f	t	f	f	t
t	f	t	f	t	t	t
f	t	f	f	t	t	t
f	f	t	f	t	t	t

Exercise 5: Show that " $P \Rightarrow [Q \vee R]$ " and " $[\tilde{}Q \wedge P] \Rightarrow R$ " are equivalent.

PROBLEM SET 3

1. Examine the truth tables of the formulas of the last two problem sets and list those formulas that are tautologies.

In each of Problems 2 - 9, determine whether the given formula is a tautology.

2. $\sim P \Leftrightarrow \sim[\sim P]$

3. $[P \wedge Q] \Rightarrow [P \vee Q]$

4. $[P \vee Q] \Rightarrow P$

5. $[P \wedge Q] \Leftrightarrow [\sim Q \Rightarrow P]$

6. $\sim[\sim P] \Leftrightarrow P$

7. $P \Leftrightarrow P$

8. $[P \Leftrightarrow Q] \Rightarrow [Q \Leftrightarrow P]$

9. $[[P \Leftrightarrow Q] \wedge [Q \Leftrightarrow R]] \Rightarrow [P \Leftrightarrow R]$

In each of Problems 10 - 21, show that the two given formulas are equivalent.

10. $\sim[P \wedge Q]$; $\sim P \vee \sim Q$

11. $\sim[P \vee Q]$; $\sim P \wedge \sim Q$

12. $\sim[P \Rightarrow Q]$; $P \wedge \sim Q$

13. $P \Rightarrow Q$; $\sim P \vee Q$

14. $P \wedge Q$; $Q \wedge P$

15. $P \vee Q$; $Q \vee P$

16. $[P \wedge Q] \wedge R$; $P \wedge [Q \wedge R]$

17. $[P \vee Q] \vee R$; $P \vee [Q \vee R]$

18. $P \Rightarrow Q$; $\sim Q \Rightarrow \sim P$

19. $P \Rightarrow [Q \Rightarrow R]$; $[P \wedge Q] \Rightarrow R$

20. $P \wedge [Q \vee R]$; $[P \wedge Q] \vee [P \wedge R]$

21. $P \vee [Q \vee R]$; $\sim[P \vee Q] \Rightarrow R$

22. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are equivalent formulas and $\mathcal{S}$ is a tautology. What can you say about $\mathcal{T}$? Explain.

23. Suppose that $\mathcal{S} \Leftrightarrow \mathcal{T}$ is a tautology. What can you say about $\mathcal{S} \Rightarrow \mathcal{T}$? $\mathcal{T} \Rightarrow \mathcal{S}$? Explain.

24. Suppose that $\mathcal{T}$ is a tautology and $\mathcal{S}$ is a well formed formula. What can you say about $\mathcal{S} \Rightarrow \mathcal{T}$? Explain.

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1.8 Some Useful Equivalences; The Substitution Principle

Let us now turn our attention to some specific equivalences that will be important to us in the work ahead:

$\sim[\sim P] \Leftrightarrow P$		Law of Double Negation
$\sim[P \wedge Q] \Leftrightarrow [\sim P \vee \sim Q]$		De Morgan's Law for Conjunction
$\sim[P \vee Q] \Leftrightarrow [\sim P \wedge \sim Q]$		De Morgan's Law for Disjunction
$\sim[P \Rightarrow Q] \Leftrightarrow [P \wedge \sim Q]$		Law of Negation of the Conditional

The above four formulas are tautologies that can be thought of as telling us how negation interacts with itself, conjunction, disjunction, and the conditional.

$[P \wedge Q] \Leftrightarrow [Q \wedge P]$	Commutative Law for Conjunction
$[P \vee Q] \Leftrightarrow [Q \vee P]$	Commutative Law for Disjunction
$[[P \wedge Q] \wedge R] \Leftrightarrow [P \wedge [Q \wedge R]]$	Associative Law for Conjunction
$[[P \vee Q] \vee R] \Leftrightarrow [P \vee [Q \vee R]]$	Associative Law for Disjunction
$[P \wedge [Q \vee R]] \Leftrightarrow [[P \wedge Q] \vee [P \wedge R]]$	Distributive Law of Conjunction over Disjunction
$[P \vee [Q \wedge R]] \Leftrightarrow [[P \vee Q] \wedge [P \vee R]]$	Distributive Law of Disjunction over Conjunction

The above six formulas are tautologies that give the basic properties of conjunction and disjunction. Finally we mention two other tautologies.

$[P \Rightarrow Q] \Leftrightarrow [\sim Q \Rightarrow \sim P]$	Law of Contraposition
$[P \Rightarrow Q] \Leftrightarrow [\sim P \vee Q]$	Law of Equivalence for the Conditional

The name of the first of these derives from the following piece of terminology: If $\mathcal{S}$ and $\mathcal{T}$ are well formed formulas, then the formula $\sim\mathcal{T} \Rightarrow \sim\mathcal{S}$ is called the <u>contrapositive</u> of $\mathcal{S} \Rightarrow \mathcal{T}$.

A word of warning! Some people are tempted to confuse equivalent formulas, especially formulas involving negation, and to say, for example, that "$\sim[\sim P]$" <u>is</u> "P". But "$\sim[\sim P]$" is not "P" for the first formula has two "$\sim$"'s in it; the second does not. No, the negation of "P" is "$\sim P$" and the negation of "$\sim P$" is "$\sim[\sim P]$", not "P". Similarly, the contrapositive of "$P \Rightarrow Q$" is "$\sim Q \Rightarrow \sim P$" and the contrapositive of "$\sim Q \Rightarrow \sim P$" is "$\sim[\sim P] \Rightarrow \sim[\sim Q]$", not "$P \Rightarrow Q$".

This would seem to be a rather unfortunate circumstance. Suppose that, for some reason, a person were compelled to take contrapositives of contrapositives, then strings of "~"'s would be generated and these are not particularly attractive. As we will see later, when we want a certain formula, an equivalent formula will suffice for most purposes. Thus, for example, if it is the contrapositive of " $\sim Q \Rightarrow \sim P$ " that is needed, a formula equivalent to the contrapositive will probably do. Thus while " $P \Rightarrow Q$ " is not the contrapositive of " $\sim Q \Rightarrow \sim P$ " it will do for certain purposes because it is equivalent to the contrapositive, which is " $\sim[\sim P] \Rightarrow \sim[\sim Q]$ ".

On what basis do we claim that " $P \Rightarrow Q$ " and " $\sim[\sim P] \Rightarrow \sim[\sim Q]$ " are equivalent? We could establish our claim by forming the truth table of the biconditional of the two given formulas and showing that the biconditional is a tautology. But that isn't necessary; there is an easier way. Let us change the problem slightly to simplify the discussion.

Consider the two formulas " $P \Rightarrow Q$ " and " $\sim[\sim P] \Rightarrow Q$ ". We know without forming the truth table that these formulas are equivalent for the following reason. Think of the second formula as having been obtained from the first by substituting " $\sim[\sim P]$ " for "P". Since " $\sim[\sim P]$ " and "P" are equivalent they have the same truth value whatever truth value is assigned to "P". Consequently, whatever the truth values of "P" and "Q", " $\sim[\sim P] \Rightarrow Q$ " has the same truth value as " $P \Rightarrow Q$ ".

Consider another example:

$$[P \vee Q] \Rightarrow Q; \qquad [Q \vee P] \Rightarrow Q.$$

Here again the second formula can be thought of as having been obtained from the first by substituting " $Q \vee P$ " for " $P \vee Q$ ". Since " $Q \vee P$ " and " $P \vee Q$ " are equivalent, the two given formulas are equivalent.

One final observation before we sum up. It isn't necessary that these substitutions be made for all occurrences of the equivalent formula. For example, " $[P \wedge Q] \Rightarrow P$ " and " $[\sim[\sim P] \wedge Q] \Rightarrow P$ " are equivalent; " $[P \wedge Q] \Rightarrow P$ " and " $[P \wedge Q] \Rightarrow \sim[\sim P]$ " are equivalent; " $[P \wedge Q] \Rightarrow P$ " and " $[\sim[\sim P] \wedge Q] \Rightarrow \sim[\sim P]$ " are equivalent.

We now summarize these observations as a basic principle.

The Substitution Principle: If $\mathcal{S}$ and $\mathcal{R}$ are equivalent formulas and $\mathcal{G}$ is a formula obtained from a formula $\mathcal{F}$ by substituting $\mathcal{R}$ for $\mathcal{S}$ in zero, one, or more occurrences of $\mathcal{S}$ in $\mathcal{F}$, then $\mathcal{G}$ and $\mathcal{F}$ are also equivalent formulas.

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Aftermath...

"Pull over to the side of the road."
"Oh, hello Officer! I'm not going to pay any attention to what you don't say."
"What?"
"Ha, ha! That was the old double negative, Officer. I'm not going to pay any attention to what you don't say, but I am going to pay attention to what you do say. Get it? Ha, ha!"
"You were doing seventy."
"It isn't your business how fast I wasn't driving."
"Say, ..."
"Ha, ha! That was the old double negative again, Officer. It isn't your business how fast I wasn't driving, but it is your business how fast I was driving. Get it? Ha, ha!"
"Well, cops enjoy a laugh as much as anybody. My, yes! I'm not going to give you a ticket for not speeding. Get it?"
He got it.

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PROBLEM SET 4

In Problems 1 - 3 write the contrapositive of the given formula.

1. $P \Rightarrow {\sim}Q$

2. ${\sim}P \Rightarrow {\sim}Q$

3. $[P \vee {\sim}Q] \Rightarrow [P \wedge Q]$

In Problems 4 - 6 show that the given formula is a tautology.

4. $[P \wedge [Q \wedge R]] \Leftrightarrow [[P \wedge Q] \wedge [P \wedge R]]$ Self-distributive Law for Conjunction

5. $[P \vee [Q \vee R]] \Leftrightarrow [[P \vee Q] \vee [P \vee R]]$ Self-distributive Law for Disjunction

6. $[P \Rightarrow [Q \Rightarrow R]] \Leftrightarrow [[P \Rightarrow Q] \Rightarrow [P \Rightarrow R]]$ Self-distributive Law for the Conditional

7. Suppose you know that " $P \Rightarrow Q$ " is false. What can you say about the truth or falsity of " ${\sim}Q \Rightarrow {\sim}P$ " ? Why?

8. If $\mathfrak{F}$ and $\mathfrak{G}$ are formulas which are related as described in the statement of the Substitution Principle and $\mathfrak{F}$ is a tautology, what can you say about $\mathfrak{G}$? Why?

9 - 11. Write the converse of each of the formulas in Problems 1 - 3.

12. In each of the following, use the Substitution Principle to show that the two given formulas are equivalent. Do this by identifying formulas $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{R}$, and $\mathfrak{S}$, and indicating where it has been shown that $\mathfrak{R}$ and $\mathfrak{S}$ are equivalent. (In (c) and (d) several applications of the Substitution Principle are necessary.)

a) $P \wedge [Q \Rightarrow R]$; $P \wedge [{}^{\sim}Q \vee R]$

b) $[P \wedge Q] \Rightarrow [R \vee {}^{\sim}P]$; $[{}^{\sim}[{}^{\sim}P] \wedge Q] \Rightarrow [R \vee {}^{\sim}P]$

c) $[R \Rightarrow S] \Rightarrow [{}^{\sim}[R \Rightarrow S] \vee [P \wedge {}^{\sim}Q]]$; $[{}^{\sim}S \Rightarrow {}^{\sim}R] \Rightarrow [[R \wedge {}^{\sim}S] \vee [P \wedge {}^{\sim}Q]]$

d) $[S \wedge [P \Rightarrow {}^{\sim}S]] \vee [[{}^{\sim}P \wedge {}^{\sim}Q] \wedge {}^{\sim}[{}^{\sim}S]]$; $[{}^{\sim}[{}^{\sim}S] \wedge {}^{\sim}[P \wedge S]] \vee [{}^{\sim}[P \vee Q] \wedge S]$

13. In each of the following, use the Substitution Principle and the result of Problem 8 to show that the given formula $\mathfrak{G}$ is a tautology. In each case, identify formulas $\mathfrak{F}$, $\mathfrak{R}$, and $\mathfrak{S}$, and indicate where it has been shown that $\mathfrak{F}$ is a tautology and $\mathfrak{R}$ is equivalent to $\mathfrak{S}$.

a) ${}^{\sim}[{}^{\sim}P] \Rightarrow [P \vee Q]$

b) $[{}^{\sim}Q \Rightarrow {}^{\sim}P] \vee [{}^{\sim}Q \vee P]$

c) ${}^{\sim}[{}^{\sim}P \vee {}^{\sim}Q] \Rightarrow P$

d) ${}^{\sim}P \Rightarrow [{}^{\sim}Q \Rightarrow {}^{\sim}P]$

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1.9 Instantiation of Formulas by Formulas; The Tautology Principle

Earlier we noted that every formula is a pattern for sentences. Every formula is also a pattern for formulas. For example, the formula

(1) $P \Rightarrow P$

is a pattern for formulas, such as the following, that are the conditional of a formula with itself:

(2) $\sim P \Rightarrow \sim P$

(3) $Q \Rightarrow Q$

(4) $[P \wedge Q] \Rightarrow [P \wedge Q]$

Each such formula we call an instance of formula (1). So now we have two kinds of instances: sentences that are instances and formulas that are instances. But no confusion will arise because the context will always make clear what we mean. Let us formulate this idea in more general terms.

Let $\mathcal{S}$ be a wff[†] and $\mathcal{P}$ be an atomic formula (i.e., a propositional variable such as "P" or "Q", etc.). If every occurrence of $\mathcal{P}$ in $\mathcal{S}$ is replaced by a formula $\mathcal{T}$, the resulting formula is an instance of $\mathcal{S}$. Or, even more generally, let $\mathcal{S}$ be a wff and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be distinct atomic formulas. If every occurrence of $\mathcal{P}_1$ in $\mathcal{S}$ is replaced by a formula $\mathcal{T}_1$, every occurrence of $\mathcal{P}_2$ in $\mathcal{S}$ is replaced by a formula $\mathcal{T}_2$, and so on, the resulting formula is an instance of $\mathcal{S}$.

For example, if $\mathcal{S}$ is " $P \Rightarrow P$ " and $\mathcal{T}$ is " $\sim P$ " and we replace each occurrence of "P" in " $P \Rightarrow P$ " by " $\sim P$ ", we obtain

$$\sim P \Rightarrow \sim P$$

which is an instance of " $P \Rightarrow P$ ".

If we replace each occurrence of "P" in " $P \Rightarrow P$ " by "Q", we obtain the instance (of " $P \Rightarrow P$ ")

$$Q \Rightarrow Q.$$

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† "wff" abbreviates "well-formed formula". (See page 7.)

If in " $P \Rightarrow Q$ " we replace "P" by "R" and "Q" by " $P \wedge Q$ ", we obtain the instance (of " $P \Rightarrow Q$ ")

$$R \Rightarrow [P \wedge Q].$$

Note that in the above description of "instances" we did not insist that each of the atomic formulas $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ occur in $\mathcal{S}$. If none of them does occur in $\mathcal{S}$, then replacement produces $\mathcal{S}$ itself as an instance of $\mathcal{S}$. Why would anyone in their right mind want to talk about replacing an atomic formula that isn't there? Not because it is important for the propositional calculus but simply because it enables us to talk about replacement in a certain way. For example, if we replace each occurrence of "P" in $\mathcal{S} \Rightarrow \mathcal{T}$, the formula we get is the conditional of the two formulas obtained by replacing each occurrence of "P" in $\mathcal{S}$ and in $\mathcal{T}$. This simple description is much more complicated if we insist that, before we can talk about replacement, "P" must actually be present.

Now that we have the notion of one formula being an instance of a second formula we are ready to describe our next logical principle, one dealing with tautologies. Consider any tautology, for example

(5) $\qquad P \Rightarrow P$

and think about its instances, say

(6) $\qquad {}^{\sim}Q \Rightarrow {}^{\sim}Q$

(7) $\qquad [P \wedge Q] \Rightarrow [P \wedge Q]$

(8) $\qquad [{}^{\sim}P \vee P] \Rightarrow [{}^{\sim}P \vee P]$.

Each of these instances is also a tautology. Must we form their truth tables to be convinced of this? No, we need only note that in (6), say, " ${}^{\sim}Q$ " has replaced "P" everywhere that "P" occurs in (5). Hence, if "Q" has truth value t, the truth value of (6) is the same as the truth value of (5) when "P" has truth value f, and vice versa. Since (5) has truth value t regardless of the truth value of "P", it follows that (6) must also have truth value t regardless of the truth value of "Q". Said another way, if (6) were not a tautology, then for some assignment of truth value to "Q" (6) would have truth value f. But then that would mean that (5) would have truth value f when "P" was assigned the opposite truth value to this particular truth value of "Q", an impossibility since (5) is a tautology.

We summarize these observations in the following principle:

The Tautology Principle: If $\mathcal{S}$ is a tautology, then every formula which is an instance of $\mathcal{S}$ is also a tautology.

For example, from the Tautology Principle we see that

$$[[P \vee Q] \wedge Q] \Rightarrow [P \vee Q]$$

is a tautology because it is an instance of the tautology " $[P \wedge Q] \Rightarrow P$ ". Combining the Tautology Principle and the Substitution Principle we see that

$$[[P \wedge Q] \wedge [\sim P \vee \sim Q]] \Rightarrow Q$$

is a tautology because it is equivalent to

$$[[P \wedge Q] \wedge \sim [P \wedge Q]] \Rightarrow Q$$

which is an instance of the tautology " $[P \wedge \sim P] \Rightarrow Q$ ".

PROBLEM SET 5

In each of Problems 1 - 5, generate an instance of the given formula by replacing "P" by " $P \vee Q$ ".

Example: Formula: $P \vee S$

Instance: $[P \vee Q] \vee S$.

1. $\sim P \wedge S$
2. $P \Rightarrow Q$
3. $\sim R \Rightarrow [S \vee R]$
4. $[P \wedge S] \Rightarrow [R \vee \sim[\sim P]]$
5. P
6. Is " $[P \vee R] \Rightarrow \sim Q$ " an instance of " $[P \vee R] \Rightarrow \sim P$ " obtained by replacing "P" by "Q" ? Explain.
7. Is " $R \wedge Q$ " an instance of " $R \wedge [S \Rightarrow P]$ " ? Explain.
8. Is " $R \wedge [S \Rightarrow P]$ " an instance of " $R \wedge Q$ " ? Explain.
9. Is " $S \Rightarrow S$ " an instance of " $R \Rightarrow S$ " ? Explain.

10. Is "$R \Rightarrow S$" an instance of "$S \Rightarrow S$"? Explain.

In each of Problems 11 - 20 use the Tautology Principle to show that the given formula is a tautology. Do this by showing that each formula is an instance of a "simpler" tautology and indicating which replacement has produced the given instance.

11. $[[R \Rightarrow S] \wedge Q] \vee \sim[[R \Rightarrow S] \wedge Q]$

12. $[T \vee \sim R] \Leftrightarrow \sim[\sim[T \vee \sim R]]$

13. $\sim[[Q \vee R] \wedge \sim[Q \vee R]]$

14. $\sim[[R \Rightarrow P] \wedge Q] \Leftrightarrow [\sim[R \Rightarrow P] \vee \sim Q]$

15. $\sim[[R \vee S] \Rightarrow Q] \Leftrightarrow [[R \vee S] \wedge \sim Q]$

16. $[[Q \Rightarrow R] \wedge S] \Rightarrow [[Q \Rightarrow R] \wedge S]$

17. $[[R \vee S] \wedge [T \Rightarrow R]] \Rightarrow [R \vee S]$

18. $[R \Rightarrow [Q \vee T]] \Rightarrow [[R \Rightarrow [Q \vee T]] \vee Q]$

19. $\sim[[R \wedge \sim S] \vee [Q \Rightarrow T]] \Rightarrow [\sim[R \wedge \sim S] \wedge \sim[Q \Rightarrow T]]$

20. $[[R \wedge P] \Rightarrow \sim Q] \Rightarrow [\sim[\sim Q] \Rightarrow \sim[R \wedge P]]$

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See Appendix I for material that will help you review for the test on Chapter 1.