2.3.1. **Quadrupole moments** (7 pts for parts a) - d) plus 3 pts for part e))

a) Consider a localized charge density as in ch.2 §3.1 and carry the expansion of the potential to $O(1/r^3)$. Show that the potential to that order is given by

$$\varphi(\boldsymbol{x}) = \frac{1}{r}Q + \frac{1}{r^3}\boldsymbol{x}\cdot\boldsymbol{d} + \frac{1}{r^5}\sum_{i,j} x_i x_j Q_{ij} + \ldots$$

with Q the total charge and $\boldsymbol{d}$ the dipole moment, and determine the quadrupole tensor Q_{ij}.

b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

c) Consider a homogeneously charged ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$ and calculate the quadrupole tensor Q_{ij} with respect to the ellipsoid's center. Check to make sure that the result for Q_{ij} is traceless.

d) Let the charge density be invariant under rotations about the z-axis through multiples of an angle α, with $|\alpha| < \pi$. Show that in this case the quadrupole tensor has the form $\begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}$. Make sure your result from part c) conforms with this for the special case $a = b$.

e) Consider the homogeneously charged ellipsoid from Part c), and calculate the quadrupole moments Q_{2m} as defined in ch.2 §3.5.

(7 points)

2.3.1.) a) $\frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{r}\left(1 - 2\frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{y^2}{r^2}\right)^{-1/2} = \frac{1}{r}\left(1 + \frac{\vec{x}\cdot\vec{y}}{r^2} - \frac{1}{2}\frac{y^2}{r^2} + \frac{3}{2}\frac{(\vec{x}\cdot\vec{y})^2}{r^4} + \dots\right)$

$$= \frac{1}{r}\left[1 + \frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{3}{2}x_i x_j y_i y_j \frac{1}{r^4} - \frac{1}{2}y^2 \delta_{ij} x_i x_j \frac{1}{r^4} + \dots\right]$$

$$= \frac{1}{r}\left[1 + \frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{1}{2}x_i x_j (3y_i y_j - \delta_{ij} y^2)\frac{1}{r^4} + \dots\right]$$

$$\rightsquigarrow \underline{\underline{\varphi(\vec{x})}} = \int d\vec{y}\, \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|} = \frac{1}{r}\int d\vec{y}\, \rho(\vec{y}) + \frac{1}{r^3}\vec{x}\cdot\int d\vec{y}\, \vec{y}\rho(\vec{y})$$

$$+ \frac{1}{2}\frac{1}{r^5} x_i x_j \int d\vec{y}\, (3y_i y_j - \delta_{ij} y^2)\rho(\vec{y}) + \dots$$

$$\underline{\underline{= \frac{1}{r}Q + \frac{1}{r^3}\vec{x}\cdot\vec{d} + \frac{1}{r^5}\sum_{ij} x_i x_j Q_{ij} + O(1/r^4)}}$$

where $Q = \int d\vec{y}\, \rho(\vec{y})$ monopole moment

$\vec{d} = \int d\vec{y}\, \vec{y}\rho(\vec{y})$ dipole moment

$\underline{\underline{Q_{ij} = \frac{1}{2}\int d\vec{y}\, (3y_i y_j - \delta_{ij} y^2)\rho(\vec{y})}}$ quadrupole moment

1

b) $\rho'(\vec{y}) = \rho(\vec{y}-\vec{a})$

$$\rightsquigarrow \underline{\underline{Q'_{ij}}} = \frac{1}{2}\int d\vec{y}\, (3y_i y_j - \delta_{ij} y^2)\rho'(\vec{y})$$

$$= \frac{1}{2}\int d\vec{y}\left[3(y_i + a_i)(y_j + a_j) - \delta_{ij}(\vec{y}+\vec{a})^2\right]\rho(\vec{y})$$

$$= Q_{ij} + \frac{1}{2}\int d\vec{y}\left[3a_i y_j + 3a_j y_i + 3a_i a_j - \delta_{ij}(2\vec{a}\cdot\vec{y} + \vec{a}^2)\right]\rho(\vec{y})$$

$$= Q_{ij} + \frac{3}{2}a_i d_j + \frac{3}{2}a_j d_i + \frac{3}{2}a_i a_j Q - \delta_{ij}\vec{a}\cdot\vec{d} - \delta_{ij}\frac{1}{2}\vec{a}^2 Q$$

1

$$= \underline{\underline{Q_{ij}}} \text{ if } \underline{\vec{d} = Q = 0}$$

c) ellipsoid: $x^2/a^2+y^2/b^2+z^2/c^2 \le 1$

$$\rightarrow \quad Q_{ij} = \frac{1}{2}\int d\vec{x}\,\left(3x_ix_j - \delta_{ij}\vec{x}^2\right)\,\Theta\left(x^2/a^2+y^2/b^2+z^2/c^2 \le 1\right)\,\rho$$

where $\vec{x} = (x, y, z)$ and ρ = charge density

(1) symmetry $\rightarrow$ $\underline{Q_{ij} = 0 \text{ unless } i=j}$

$$\underline{\underline{Q_{11}}} = \frac{\rho}{2}\int dx\,dy\,dz\,\left(2x^2-y^2-z^2\right)\,\Theta\left(x^2/a^2+y^2/b^2+z^2/c^2 \le 1\right)$$

$$= \frac{1}{2}\rho abc\int dx\,dy\,dz\,\left(2a^2x^2-b^2y^2-c^2z^2\right)\,\Theta\left(x^2+y^2+z^2 \le 1\right)$$

$$= \frac{1}{2}\rho abc\Big[2a^2\int_0^1 dr\,r^2\int_{-1}^{1} d\eta\int_0^{2\pi} d\varphi\, r^2\sin^2\vartheta\cos^2\varphi$$

$$-b^2\int_0^1 dr\,r^2\int_{-1}^{1} d\eta\int_0^{2\pi} d\varphi\, r^2\sin^2\vartheta\sin^2\varphi$$

$$-c^2\int_0^1 dr\,r^2\int_{-1}^{1} d\eta\int_0^{2\pi} d\varphi\, r^2\cos^2\vartheta\Big]$$

$$= \frac{1}{2}\rho abc\left[2a^2\pi\left(2-\frac{2}{3}\right)\frac{1}{5} - b^2\pi\left(2-\frac{2}{3}\right)\frac{1}{5} - c^2 2\pi\,\frac{2}{3}\frac{1}{5}\right]$$

$$= \frac{1}{2}\rho abc\,\frac{\pi}{5}\left[\frac{8}{3}a^2 - \frac{4}{3}b^2 - \frac{4}{3}c^2\right]$$

$$= \frac{4\pi}{3}abc\rho\,\frac{1}{10}\left[2a^2-b^2-c^2\right]$$

(1) $$= \underline{Q\,\frac{1}{10}\left(2a^2-b^2-c^2\right)}$$ with $\underline{Q = \frac{4\pi}{3}\rho abc}$ = total charge

$\underline{\underline{Q_{22}}} = \underline{Q\,\frac{1}{10}\left(2b^2-a^2-c^2\right)}$ by symmetry

$\underline{\underline{Q_{33}}} = \underline{Q\,\frac{1}{10}\left(2c^2-b^2-a^2\right)}$ by symmetry

(1) check: $\underline{Q_{11}+Q_{22}+Q_{33} = 0}$ ✓

d) As a real symmetric tensor, Q_{ij} can always be diagonalized

→ The most general form of Q_{ij} in its principal axes system is

$$Q_{ij} = \begin{pmatrix} q_+ + q_- & 0 & 0 \\ 0 & q_+ - q_- & 0 \\ 0 & 0 & -2q_+ \end{pmatrix}$$

where

$$\underline{q_-} = \frac{1}{2}\left(Q_{11} - Q_{22}\right) = \frac{1}{2}\int d\vec{x}\,\rho(\vec{x})\left[2x^2 - y^2 - z^2 - 2y^2 + x^2 + z^2\right]$$

$$= \underline{\frac{3}{2}\int d\vec{x}\,\rho(\vec{x})\left(x^2 - y^2\right)}$$

(1)

cylindrical coordinates: $x = r\cos\varphi$, $y = r\sin\varphi$ $\qquad x^2 - y^2 = r^2(\cos^2\varphi - \sin^2\varphi) = r^2\cos 2\varphi$

$$\rightarrow \quad q_- = \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r \int dz\, \rho(r,\varphi,z)\, r^2 \cos 2\varphi$$

Now let $\rho(r,\varphi,z) = \rho(r,\varphi+\alpha,z)$

$$\rightarrow \quad q_- = \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi+\alpha,z)\cos 2\varphi$$

$$= \frac{3}{2}\int_\alpha^{2\pi+\alpha} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi,z)\cos 2(\varphi - \alpha)$$

$$= \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi,z)\left(\cos 2\varphi\cos 2\alpha + \sin 2\varphi \sin 2\alpha\right)$$

$\boxed{r^2\sin 2\varphi = 2r^2\sin\varphi\cos\varphi = xy \quad \rightarrow \text{ the second term is } \propto Q_{12} = 0}$

$$= \cos 2\alpha \cdot q_- \quad \rightarrow \quad \underline{\underline{q_- = 0}} \text{ since } \alpha \neq \pi$$

(1)

part c) with $a = b$ → $Q_{11} = Q_{22} = \frac{Q}{10}(a^2 - c^2)$ ✓

p-3 JJ I-4

e.) $\underline{\underline{Q_{20}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega)\, \cancel{\sqrt{\frac{5}{4\pi}}}\, \frac{1}{2}(3\zeta^2-1)$

$= \frac{1}{2}\int d\vec{x}\, \rho(\vec{x})\,(3z^2-r^2) = \underline{\underline{Q_{33}}}$

(1)

$\underline{\underline{Q_{2,\pm 1}}} \propto \int d\Omega\, \underbrace{\underbrace{\rho(r,\Omega)}_{\text{even}}\, \underbrace{P_2^{\pm 1}(\zeta)}_{\text{odd}}}_{\text{fct. of } \zeta} \;\underline{\underline{= 0}}$ by symmetry

(1)

$\underline{\underline{Q_{22}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega)\, \cancel{\sqrt{\frac{5}{4\pi}}}\, \frac{1}{\sqrt{4!}}\, e^{2i\varphi}\, 3(1-\zeta^2)$

$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 (1-\zeta^2)(\cos 2\varphi + i \sin 2\varphi)$

$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 \sin^2\vartheta (\cos^2\varphi - \sin^2\varphi) + \frac{3i}{\sqrt{24}} \int d\vec{x}\, \underbrace{\underbrace{\rho(\vec{x})}_{\text{even}}\, r^2 \sin^2\vartheta\, \underbrace{\sin 2\varphi}_{\text{odd}}}_{\text{fct. of } \varphi}$ $\to 0$

$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 (\sin^2\vartheta \cos^2\varphi - \sin^2\vartheta \sin^2\varphi)$

$$\left.\begin{aligned} x &= r \sin\vartheta \cos\varphi \\ y &= r \sin\vartheta \sin\varphi \\ z &= r\cos\vartheta \end{aligned}\right\} \rightarrow x^2 - y^2 = r^2 \sin^2\vartheta (\cos^2\varphi - \sin^2\varphi)$$

$= \frac{3}{2\sqrt{6}} \int d\vec{x}\, \rho(\vec{x})\, (x^2-y^2)$

$= \frac{3}{2\sqrt{6}} \int d\vec{x}\, \rho(\vec{x}) \left[(2x^2-y^2-z^2) - (2y^2-x^2-z^2)\right]\frac{1}{3}$

$= \underline{\underline{\frac{1}{\sqrt{6}}(Q_{11} - Q_{22})}}$

Q

$\underline{\underline{Q_{2,-2}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega)\, \cancel{\sqrt{\frac{5}{4\pi}}}\, \sqrt{4!}\, e^{-2i\varphi}\, \frac{1}{8}(1-\zeta^2) = \underline{\underline{Q_{22}}}$

(1)