1.4.3 **Lorentz transformations of fields**

Consider static and homogeneous fields $\boldsymbol{E}$ and $\boldsymbol{B}$ that are not parallel to one another in some inertial frame.

a) Show that there exists an inertial frame in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel, and that the two frames are related by a Lorentz boost whose velocity is given by the solution of the equation

$$\frac{\boldsymbol{V}}{c}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) = (1 + \boldsymbol{V}^2/c^2)\,\boldsymbol{E} \times \boldsymbol{B}$$

b) Show explicitly that this equation has one and only one physical solution that obeys $|\boldsymbol{V}|/c < 1$, that there always is a physical solution, and that the result in the limit of almost parallel fields in the original reference frame is sensible.

c) Are there other inertial frames in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel? If so, how many?

(7 points)

1.4.3.) a) Let the non-parallel vectors $\vec{E}, \vec{B}$ lie in the y-z-plane:

$$\vec{E} = (0, E_y, E_z)\,, \quad \vec{B} = (0, B_y, B_z)$$

Now consider a Lorentz boost in the x-direction

cf 2 §4.2 $\rightsquigarrow$ $\hat{E}_x = E_x = 0$ and $\hat{B}_x = B_x = 0$.

$$\rightsquigarrow \quad \hat{\vec{E}} = (0, \hat{E}_y, \hat{E}_z)\,, \quad \hat{\vec{B}} = (0, \hat{B}_y, \hat{B}_z)$$

$$\rightsquigarrow \quad \hat{\vec{E}} \times \hat{\vec{B}} = (\hat{E}_y \hat{B}_z - \hat{E}_z \hat{B}_y, 0, 0)$$

0

$$\rightsquigarrow \quad \hat{\vec{E}} \parallel \hat{\vec{B}} \quad \text{iff} \quad \hat{E}_y \hat{B}_z - \hat{E}_z \hat{B}_y = 0$$

1

cf 2 §4.2 $\rightsquigarrow$

$$\underline{0} = (E_y \cosh\phi + B_z \sinh\phi)(B_z \cosh\phi + E_y \sinh\phi) - (E_z \cosh\phi - B_y \sinh\phi)(B_y \cosh\phi - E_z \sinh\phi)$$

$$= (E_y B_z - E_z B_y)(\cosh^2\phi + \sinh^2\phi) + (E_y^2 + E_z^2 + B_y^2 + B_z^2)\cosh\phi \sinh\phi$$

$$\overset{\S 4.1}{=} \underline{\gamma^2 (1+\beta^2)(\vec{E} \times \vec{B})_x + \gamma^2 \beta (\vec{E}^2 + \vec{B}^2)} \qquad \text{where } \beta = V/c$$

0

If we make the orientation of the plane defined by $\vec{E}$ and $\vec{B}$ arbitrary, then $\vec{V}$ is given by the solution of

1

$$\underline{\underline{\frac{\vec{V}}{c}(\vec{E}^2 + \vec{B}^2) = (1 + \vec{V}^2/c^2)\, \vec{E} \times \vec{B}}}$$

b) Now return to the original geometry and denote

$$(\vec{E} \times \vec{B})_x =: X\,, \quad \vec{E}^2 + \vec{B}^2 =: S\,, \quad V_x =: V$$

$$\rightsquigarrow \quad X\beta^2 - S\beta + X = 0$$

$\rightsquigarrow \quad \Lambda = \frac{1}{2x}\left(s \pm \sqrt{s^2-4x^2}\right) \qquad (*)$

Now consider $s^2-4x^2 = (E^2+B^2)^2 - 4E^2B^2\sin^2\alpha \qquad \alpha = \angle(\vec{E},\vec{B})$

$$\geq (E^2+B^2)^2 - 4E^2B^2$$

$$= (E^2-B^2)^2 \geq 0 \qquad (+)$$

$\rightsquigarrow \sqrt{s^2-4x^2} \in \mathbb{R}$ always

(1) $\rightsquigarrow$ There always is at least one solution

That $(+) \rightsquigarrow \left(\frac{s}{2x}\right)^2 \geq 1$

Let $x > 0$ w.l.g. $\rightsquigarrow \quad s/2x > 1$

$$\rightsquigarrow \frac{s}{2x} + \frac{1}{2x}\sqrt{s^2-4x^2} > 1$$

$\rightsquigarrow$ The only candidate for a physical solution is

(1) $\Lambda = \frac{1}{2x}\left(s-\sqrt{s^2-4x^2}\right) = \frac{s}{2x} - \sqrt{\left(\frac{s}{2x}\right)^2-1}$

We still need to show that this solution obeys $\Lambda < 1$.

Define $\alpha := s/2x > 0$ w.l.g.

Then we know $\alpha > 1$.

Now demand $\alpha - \sqrt{\alpha^2-1} \overset{!}{<} 1$

$\Leftrightarrow \alpha - 1 < \sqrt{\alpha^2-1} \Leftrightarrow (\alpha-1)^2 < \alpha^2-1 \Leftrightarrow -2\alpha+1 < -1$

$\Leftrightarrow \alpha > 1$ ✓

(1) $\rightsquigarrow \quad \Lambda = \alpha - \sqrt{\alpha^2-1}$ where $\alpha = \frac{s}{2x} = \frac{E^2+B^2}{2|\vec{E}\times\vec{B}|}$ is the unique physical solution which always exists

Consider almost parallel fields $\to$ $\vec{E}\times\vec{B}\to 0$ $\to$ $\alpha\to\infty$

$$\to \quad \underline{\underline{\Lambda}} \to \alpha-\alpha\sqrt{1-1/\alpha^2} = \alpha-\alpha+\frac{1}{2\alpha}+O(1/\alpha^3) = \frac{1}{2\alpha}+O(1/\alpha^3) \to \underline{\underline{0}}$$

which is sensible: If the fields are almost parallel to start with, then a small boost velocity suffices to make them parallel.

(1)

c) Once $\vec{E}\parallel\vec{B}$, e.g., $\vec{E}=(E_x,0,0)$, $\vec{B}=(B_x,0,0)$ w.l.o.g., then any Lorentz boost in the common direction of $\vec{E}$ and $\vec{B}$ will leave $\vec{E}\parallel\vec{B}$ $\to$ There are infinitely many such inertial frames

(1)