1.4.2. **Galileo transformations of Maxwell's equations**

a) Show explicitly which of Maxwell's equations are or are not invariant under Galileo transformations. *hint:* Consider the transformations of all vectors (the 4-gradient, the fields, and the 4-current) to zeroth order in $1/c$, but keep the terms of $O(1/c)$ in Maxwell's equations. In other words, note that if you do a Lorentz transformation consistently to a given order in $1/c$, then of course all of Maxwell's equations are invariant.

b) Suppose you had never heard of Lorentz transformations, but were familiar with Galilean mechanics. What are the two logical conclusions you could draw from the result of part a)? (Obviously, one of them by now is of historical interest only.)

(4 points)

1.4.2.) a) Consider a Lorentz boost along the x-axis (cf 2 §4.1):

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1+O(v^2/c^2) & v/c+O(v^2/c^2) & 0 & 0 \\ v/c+O(v^2/c^2) & 1+O(v^2/c^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and transform the 4-position:

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \Lambda^\mu{}_\nu \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct + O(1/c) \\ vt + x + O(1/c^2) \\ y \\ z \end{pmatrix}$$

→ To leading order in $1/c$, $\underline{\tilde{t} = t \,,\ \tilde{\vec{x}} = \vec{x} + \vec{v}t}$ ✓

Now transform the derivatives:

$$\tilde{\partial}_\mu = \begin{pmatrix} \frac{1}{c}\partial_{\tilde{t}} \\ \tilde{\vec{\nabla}} \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\partial_t + \frac{v}{c}\partial_x + \dots \\ \partial_x + \dots \\ \partial_y \\ \partial_z \end{pmatrix} \rightarrow \underline{\partial_{\tilde{t}} = \partial_t + \vec{v}\cdot\vec{\nabla} \,,\ \tilde{\vec{\nabla}} = \vec{\nabla}}$$

to $O(1/c^0)$

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The fields have no time-like component

→ $\underline{\tilde{\vec{E}} = \vec{E} \,,\ \tilde{\vec{B}} = \vec{B}}$ to $O(1/c^0)$

(this also follows explicitly from cf 2 §4.2).

Finally, the 4-current:

$$\tilde{j}^\mu = \begin{pmatrix} c\tilde{\rho} \\ \tilde{\vec{j}} \end{pmatrix} = \begin{pmatrix} c\rho + \dots \\ \vec{v}\rho + \vec{j} + \dots \end{pmatrix} \rightarrow \underline{\tilde{\rho} = \rho \,,\ \tilde{\vec{j}} = \vec{j} + \vec{v}\rho}$$

to $O(1/c^0)$

1

Now consider Maxwell's eqs.:

(1) $\underline{\hat{\vec{\nabla}} \cdot \hat{\vec{B}}} = \vec{\nabla} \cdot \vec{B} = \underline{\underline{0}}$ ✓

(2) $\underline{\frac{1}{c}\partial_{\tilde{t}} \hat{\vec{B}} + \hat{\vec{\nabla}} \times \hat{\vec{E}}} = \underbrace{\frac{1}{c}\partial_t \vec{B} + \vec{\nabla} \times \vec{E}}_{=0} + \frac{1}{c}(\vec{V} \cdot \vec{\nabla})\vec{B} = \underline{\underline{\frac{1}{c}(\vec{V} \cdot \vec{\nabla})\vec{B}}} \neq 0$

(3) $\underline{\underline{\hat{\vec{\nabla}} \cdot \hat{\vec{E}}}} = \vec{\nabla} \cdot \vec{E} = 4\pi\rho = \underline{\underline{4\pi\hat{\rho}}}$ ✓

(4) $\underline{-\frac{1}{c}\partial_{\tilde{t}} \hat{\vec{E}} + \hat{\vec{\nabla}} \times \hat{\vec{B}}} = -\frac{1}{c}\partial_t \vec{E} + \vec{\nabla} \times \vec{B} - \frac{1}{c}(\vec{V} \cdot \vec{\nabla})\vec{E}$

$$= \frac{4\pi}{c}\vec{j} - \frac{1}{c}(\vec{V} \cdot \vec{\nabla})\vec{E}$$

$$= \underline{\underline{\frac{4\pi}{c}\hat{\vec{j}} - \frac{4\pi}{c}\rho\vec{V} - \frac{1}{c}(\vec{V} \cdot \vec{\nabla})\vec{E}}} \neq \frac{4\pi}{c}\hat{\vec{j}}$$

→ <u>(1) and (3) are covariant under Galileo boosts, but (2) and (4) are not</u>

(1)

<u>remark</u>: (1) The eqs that contain a time derivative are not covariant. The "streaming term" $\vec{V} \cdot \vec{\nabla}$ in $\partial_{\tilde{t}}$ screws things up (and also the $\rho\vec{V}$ term in $\hat{\vec{j}}$, which is a velocity term and hence equivalent to a time derivative.

(2) Another way to say it: M's eqs contain terms of $O(1/c)$ in (2) and (4), whereas the Galileo boost contains the transformation only to $O(1)$, so things are not covariant.

b) Two possibilities:

(1) Maxwell's eqs are valid only in a special reference frame known as "ether". Michelson - Morley killed that possibility.

(2) Newtonian mechanics is wrong. This turned out to be the resolution; special relativity fixed the problem.

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