1.4.1. **Addition of velocities**

Consider a particle that has a velocity $\boldsymbol{v}$ in some inertial frame. Find the velocity of the particle in another inertial frame that moves with a velocity $\boldsymbol{V}$ with respect to the first one. Use the result to show that the velocity in the second frame is less than c, provided it was less than c in the first one.

(3 points)

1.4.1.) d1 §4.1 $\to$ Lorentz boost in x-direction

$$\hat{x} = \gamma x + \gamma V t \ , \quad \hat{t} = \gamma t + \gamma \frac{V}{c^2} x \qquad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$$

$$\to \quad d\hat{x} = \gamma (dx + V dt) \ , \quad d\hat{t} = \gamma \left(dt + \frac{V}{c^2} dx\right)$$

$$d\hat{y} = dy$$

$$d\hat{z} = dz$$

$$\to \underline{\underline{\hat{v}_x}} = \frac{d\hat{x}}{d\hat{t}} = \frac{d\hat{x}}{dt} \frac{1}{d\hat{t}/dt} = \frac{\gamma v_x + \gamma V}{\gamma + \gamma V v_x / c^2} = \underline{\underline{\frac{v_x + V}{1 + v_x V/c^2}}} \qquad (1)$$

$$\underline{\underline{\hat{v}_y}} = \frac{d\hat{y}}{d\hat{t}} = \frac{v_y}{\gamma (1 + v_x V/c^2)} = \underline{\underline{\frac{v_y \sqrt{1 - V^2/c^2}}{1 + v_x V/c^2}}} \qquad (2)$$

$$\underline{\underline{\hat{v}_z}} = \underline{\underline{\frac{v_z \sqrt{1 - V^2/c^2}}{1 + v_x V/c^2}}} \qquad (3)$$

(1)

let $\hat{\beta}_x = \hat{v}_x / c \ , \ \beta = V/c$

$$\to \underline{1 - \hat{\beta}_x^2} = 1 - \frac{(\beta_x + \beta)^2}{(1 + \beta_x \beta)^2} = \frac{1 + \cancel{2\beta_x\beta} + \beta_x^2 \beta^2 - \beta_x^2 - \beta^2 - \cancel{2\beta_x\beta}}{(1 + \beta_x \beta)^2} =$$

$$= \frac{(1 - \beta_x^2)(1 - \beta^2)}{(1 + \beta_x \beta)^2} \ \underline{\geq 0} \quad \text{provided } \beta_x, \beta \leq 1$$

and $1 - \hat{\beta}_x^2 = 0$ iff ($\beta_x = 1$ or $\beta = 1$)

(1) $\to \hat{\beta}_x^2 \leq 1 \to \underline{\underline{|\hat{v}_x| \leq c}}$ and $|\hat{v}_x| = c$ iff ($|v_x| = c$ or $|V| = c$)

Now define $v_\perp^2 := v_y^2 + v_z^2 \quad \to \quad v_x^2 + v_\perp^2 \leq c^2$

$$\to \quad \underline{v_\perp^2 \leq c^2 - v_x^2} \qquad (*)$$

$$(2),(3) \to \underline{\hat{v}_\perp^2} = v_\perp^2 \frac{1 - \beta^2}{(1 + \beta_x \beta)^2} \overset{(*)}{\leq} c^2 (1 - \beta_x^2) \frac{1 - \beta^2}{(1 + \beta_x \beta)^2}$$

$$\to \underline{1 - \hat{\beta}_\perp^2} := 1 - \frac{\hat{v}_\perp^2}{c^2} \geq 1 - \frac{(1 - \beta_x^2)(1 - \beta^2)}{(1 + \beta_x \beta)^2} = \frac{\cancel{1} + 2\beta_x \beta + \cancel{\beta_x^2 \beta^2} - \cancel{1} + \beta_x^2 + \beta^2 - \cancel{\beta_x^2 \beta^2}}{(1 + \beta_x \beta)^2}$$

$$= \underline{\frac{(\beta_x + \beta)^2}{(1 + \beta_x \beta)^2} > 0} \quad \to \underline{\hat{\beta}_\perp^2 < 1} \quad \to \quad \underline{\underline{|v_\perp| \leq c}}$$

(1)