1.2.5. Charged Klein-Gordon field (8 pts)

This is a continuation of Problem 0.2.5.

a) Modify the Lagrangian density for the massive scalar field, Problem 0.2.5, to read

$$\mathcal{L} = (D_\mu \phi(x)) (D^\mu \phi(x))^* - m^2 |\phi(x)|^2 - \frac{1}{16\pi} F_{\mu\nu}(x) F^{\mu\nu}(x)$$

with $D_\mu = \partial_\mu - iqA_\mu$, where A_μ is the 4-vector potential and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. Show that $\mathcal{L}$ is invariant under local gauge transformations

$$\phi(x) \to e^{iq\lambda(x)} \phi(x) \qquad , \qquad A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$$

b) Derive the field equations for the action

$$S = \int d^4x \, \mathcal{L}$$

hint: Remember $\partial\mathcal{L}/\partial(\partial_\mu A_\nu) = F^{\nu\mu}/4\pi$, which we derived in class.

Show that two of the Euler-Lagrange equations are generalizations of the Klein-Gordon equation $(\partial_\mu \partial^\mu + m^2)\phi = 0$, while the third one is the electromagnetic field equation with a particular 4-current density (which you may recognize from Quantum Mechanics).

Solution

1.2.5.) $\mathcal{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 |\phi|^2 - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$

with $D_\mu = \partial_\mu - igA_\mu$

a) $\phi(x) \to \phi(x) e^{ig\lambda(x)}$, $A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$ local gauge trafo

The various parts of $\mathcal{L}$ transform as follows:

$D_\mu \to \partial_\mu - ig(A_\mu + \partial_\mu \lambda) = D_\mu - ig\partial_\mu \lambda$, $D_\mu^* \to D_\mu^* + ig\partial_\mu \lambda$

$\to \underline{(D_\mu \phi)} \to (D_\mu - ig\partial_\mu \lambda) e^{ig\lambda} \phi = (\partial_\mu - ig(\partial_\mu \lambda) - igA_\mu) e^{ig\lambda} \phi$

$= (\partial_\mu \phi) e^{ig\lambda} + ig\cancel{(\partial_\mu \lambda)} \phi e^{ig\lambda} - ig\cancel{(\partial_\mu \lambda)} e^{ig\lambda} \phi - igA_\mu e^{ig\lambda} \phi$

$= \underline{(D_\mu \phi) e^{ig\lambda}}$

$\underline{(D_\mu \phi)^*} \to \underline{(D_\mu \phi)^* e^{-ig\lambda}}$

1 $\to \underline{(D_\mu \phi)(D_\mu \phi)^*} \to (D_\mu \phi) e^{ig\lambda} (D_\mu \phi)^* e^{-ig\lambda} = \underline{(D_\mu \phi)(D_\mu \phi)^*}$ invariant

$\underline{|\phi|^2} = \phi\phi^* \to \phi e^{ig\lambda} \phi^* e^{-ig\lambda} = \underline{|\phi|^2}$ invariant

$\underline{F_{\mu\nu}} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu + \partial_\mu \partial_\nu \lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \lambda$

$= \partial_\mu A_\nu - \partial_\nu A_\mu = \underline{F_{\mu\nu}}$

$\to F_{\mu\nu} F^{\mu\nu}$ is invariant

1 $\to$ $\underline{\mathcal{L} \text{ is invariant}}$

b) $S' = \int d^4x \left((\partial_\mu - igA_\mu)\phi \right)\left((\partial_\mu + igA_\mu)\phi^* \right) - m^2 \int d^4x\, \phi\phi^*$

$$- \frac{1}{16\pi} \int d^4x \, (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$0 \stackrel{!}{=} \delta S' = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) + c.c.$$

$$+ \int d^4x \left(\frac{\partial \mathcal{L}}{\partial A_\mu} \delta A_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \delta(\partial_\nu A_\mu) \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + c.c. + \int d^4x \left(\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) \delta A_\mu$$

Fundamental lemma ->

$$(*) \quad \boxed{\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}} \quad + c.c.$$

$$(**) \quad \boxed{\frac{\partial \mathcal{L}}{\partial A_\mu} = \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}} \qquad \underline{\text{Euler-Lagrange eqs}}$$

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Explicitly, we have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi^* - igA_\mu (D^\mu \phi)^*, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = (D^\mu \phi)^*$$

1 $\quad \frac{\partial \mathcal{L}}{\partial A_\mu} = -ig\phi (D^\mu\phi)^* + ig\phi^*(D^\mu \phi), \quad \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{1}{4\pi} F^{\mu\nu}$

-> (*) reads explicitly

$$-m^2\phi^* - igA_\mu (D^\mu\phi)^* = \partial_\mu (D^\mu \phi)^*$$

$$\rightarrow -m^2\phi^* = (\partial_\mu + igA_\mu)(D^\mu\phi)^* = D_\mu^*(D^\mu\phi)^* = (D_\mu D^\mu \phi)^*$$

1 $\quad \rightarrow \boxed{(D_\mu D^\mu + m^2)\phi = 0} \quad + c.c. \qquad (*')$

and (**) reads explicitly

$$\frac{1}{4\pi}\partial_\nu F^{\mu\nu} = -ig\phi(D^\mu\phi)^* + c.c.$$

1 $$\rightarrow \boxed{\partial_\mu F^{\mu\nu} = 4\pi i g\,\phi(D^\nu\phi)^* + c.c.} \qquad (**')$$

interpretation:

(*') is a generalization of the Klein-Gordon eq. that couples the particle to the electromagnetic field.

1 The generalization consists of $\partial_\mu \to D_\mu$.

(**') are the inhomogeneous Maxwell eqs with the 4-current given by

$$j^\mu = icg\,\phi(D^\mu\phi)^* + c.c.$$

1 $$= cg\,\phi(i\partial^\mu - gA^\mu)\phi^* + c.c.$$