1.2.1. **Energy-momentum tensor**

Consider the electromagnetic field in the absence of matter.

a) Show that the tensor field

$$H_\mu{}^\nu(x) = (\partial_\mu A_\alpha(x)) \, \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu A_\alpha(x)\right)} - \delta_\mu{}^\nu \, \mathcal{L}$$

obeys the continuity equation

$$\partial_\nu \, H_\mu{}^\nu(x) = 0 \quad (*)$$

note: Notice that $H_\mu{}^\nu(x)$ is a generalization of Jacobi's integral in Classical Mechanics.

b) Show that (*) also holds for

$$\tilde{T}_\mu{}^\nu = H_\mu{}^\nu + \partial_\alpha \psi_\mu{}^{\nu\alpha}$$

where $\psi_\mu{}^{\nu\alpha}$ is any tensor field that is antisymmetric in the second and third indices, $\psi_\mu{}^{\nu\alpha}(x) = -\psi_\mu{}^{\alpha\nu}(x)$.

c) Show that $\psi_\mu{}^{\nu\alpha}$ can be chosen such that $\tilde{T}_\mu{}^\nu(x) = T_\mu{}^\nu(x)$, which provides an alternative proof that $T_\mu{}^\nu(x)$ obeys (*).

(5 points)

1.2.1.) a) $\underline{\partial_\nu \delta_\mu{}^\nu \mathcal{L}} = \partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial A_\alpha}\partial_\mu A_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\partial_\mu \partial_\lambda A_\alpha$

(1) $\overset{EL}{=} \left(\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\right)\partial_\mu A_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\partial_\mu \partial_\lambda A_\alpha$

$= \underline{\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\partial_\mu A_\alpha}$

(1) $\rightarrow \underline{\underline{0}} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\alpha)}\partial_\mu A_\alpha - \delta_\mu{}^\nu \mathcal{L}\right) = \underline{\underline{\partial_\nu \Theta_\mu{}^\nu}}$

b) $\underline{\partial_\nu \partial_\alpha \psi_\mu{}^{\nu\alpha}} = -\partial_\nu\partial_\alpha \psi_\mu{}^{\alpha\nu} = -\partial_\alpha \partial_\nu \psi_\mu{}^{\nu\alpha} = \underline{-\partial_\nu\partial_\alpha\psi_\mu{}^{\nu\alpha}}$

(1) $\rightarrow \underline{\partial_\nu\partial_\alpha \psi_\mu{}^{\nu\alpha} = 0} \quad \rightarrow \underline{\underline{\partial_\nu \tilde{T}_\mu{}^\nu = 0}}$

(1) c) Wenn $\psi^{\mu\nu\alpha} = \frac{1}{4\pi} A^\mu F^{\nu\alpha} = \frac{-1}{4\pi} A^\mu F^{\alpha\nu} = -\psi^{\mu\alpha\nu}$ ✓

$\rightarrow \partial_\nu \tilde{T}_\mu{}^\nu = 0$, ok

$\underline{\underline{\tilde{T}^{\mu\nu}}} = \Theta^{\mu\nu} + \partial_\alpha \psi^{\mu\nu\alpha} = (\partial^\mu A_\alpha)\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\alpha)} - g^{\mu\nu}\mathcal{L} + \frac{1}{4\pi}\partial_\alpha A^\mu F^{\nu\alpha}$

$\overset{1.1}{=} (\partial^\mu A_\alpha)\frac{1}{4\pi}F^{\alpha\nu} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\lambda}F^{\alpha\lambda} + \frac{1}{4\pi}(\partial_\alpha A^\mu)F^{\nu\alpha} + \frac{1}{4\pi}A^\mu \underbrace{\partial_\alpha F^{\nu\alpha}}_{=0}$

$= \frac{-1}{4\pi}\left(\partial^\mu A_\alpha - \partial_\alpha A^\mu\right)F^{\nu\alpha} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\lambda}F^{\alpha\lambda}$

$= \frac{-1}{4\pi} F^\mu{}_\alpha F^{\nu\alpha} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\lambda}F^{\alpha\lambda}$

$= \frac{-1}{4\pi} F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\lambda}F^{\alpha\lambda} = \underline{\underline{T^{\mu\nu}}}$

(1) $\rightarrow \underline{\underline{\partial_\nu T_\mu{}^\nu = 0}}$