0.2.3 **Geodesics on the 2-sphere**

Show that the geodesics on the 2-sphere are great circles.

hint: There are various ways of doing this. One is to set up the problem of geodesics in $\mathbb{R}_3$ with the constraint that the desired paths $\vec{x}(t)$ must lie on the sphere. Now use the Euler-Lagrange equations for the constrained problem to show that $\vec{\ell} = \vec{x} \times \vec{p} = \text{const}$, where $\vec{p} = \partial L / \partial \dot{\vec{x}}$, with L the appropriate Lagrangian.

(5 points)

0.2.3.) Consider a path in $\mathbb{R}_3$: $\vec{x}(t) = (x_1(t), x_2(t), x_3(t))$

length of curve: $l = \int_{t_-}^{t_+} dt \sqrt{\dot{x}_1^2(t) + \dot{x}_2^2(t) + \dot{x}_3^2(t)}$

$$=: \int_{t_-}^{t_+} dt \; L_1(\dot{\vec{x}})$$

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Boundary condition: $x_1^2(t) + x_2^2(t) + x_3^2(t) = 1 \quad (+)$

$$0 = \int_{t_-}^{t_+} dt \left(x_1^2(t) + x_2^2(t) + x_3^2(t) - 1\right)$$

$$=: \int_{t_-}^{t_+} dt \; L_2(\vec{x})$$

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→ consider $L = L_1 - \lambda L_2$

$$L(\vec{x}, \dot{\vec{x}}) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} - \lambda\left(x_1^2 + x_2^2 + x_3^2 - 1\right)$$

Euler-Lagrange → $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}$

$$\frac{\dot{x}_i}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}} = -2\lambda x_i$$

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Now consider $\vec{l} := \vec{x} \times \vec{p}$ with $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}$

→ $\vec{p} \parallel \dot{\vec{x}}$, or $\underline{\vec{p} \times \dot{\vec{x}} = 0}$

Further, (+) → $\dot{\vec{p}} = -2\lambda \vec{x}$ → $\underline{\dot{\vec{p}} \times \vec{x} = 0}$

→ $\dot{\vec{l}} = \dot{\vec{x}} \times \vec{p} + \vec{x} \times \dot{\vec{p}} = 0$ → $\vec{l} = \text{const}$

1 and $\vec{l} \cdot \vec{x} = \vec{x} \cdot (\vec{x} \times \vec{p}) = 0$ → $\boxed{l_1 x_1 + l_2 x_2 + l_3 x_3 = 0}$ (++) with $l_{1,2,3} = \text{const}$

(++) describes a plane that contains the origin
(+) describes a sphere whose center is the origin
=> The geodesics are great circles

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