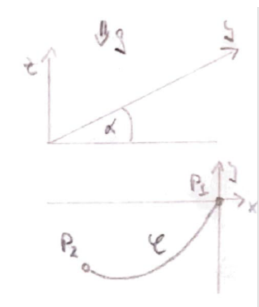
0.2.1 **The brachistrochrone problem**

A point mass glides without friction on an inclined plane (inclination angle α) from point P_1 to point P_2 on a path $\mathfrak{C}$ according to Galilean mechanics.



a) Use energy conservation to find the velocity as a function of y, using the coordinate system in the sketch.

b) Write the passage time from P_1 to P_2 in the form

$$T = \int_{x_1}^{x_2} dx\, L(y, y')$$

with y considered a function of x and $y' = dy/dx$, and determine the Lagrangian L. Use the fact that Jacobi's integral is constant to find an ODE for y.

c) Substitute $y' = \operatorname{ctg} t$ to write the brachistochrone in a parametric form, $y = y(t)$, $x = x(t)$.

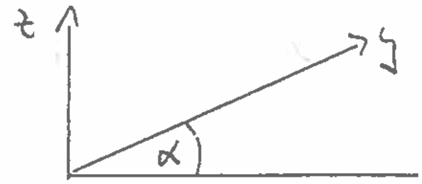
d) Express the passage time as a function of the terminal y-velocity, $y_2' = (dy/dx)_{P_2}$.

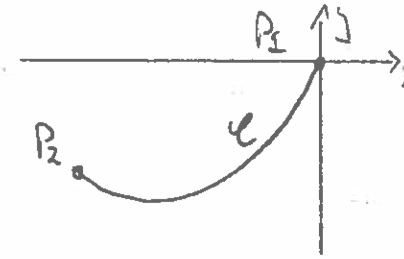
e) Find the passage time for the shortest path from P_1 to P_2 (as opposed to the brachistochrone) as a function of the same y_2'.

f) Discuss the ratio of the two passage times as a function of y_2'.

hint: The parameter value t_2 for the brachistochrone at the end point P_2 is a known function of y_2'. It therefore suffices to discuss the passage time as a function of t_2.

(18 points)





0.2.1.) constraint: $z = y \sin\alpha$

let $P_1 = (0,0,0)$ wlog

a) Energy conservation ->

$$\frac{m}{2}v^2 = -mgz = -mgy\sin\alpha$$

1 -> $v(x,y) = v(y) = \sqrt{2g\sin\alpha}\sqrt{-y} =: \sqrt{-ay}$ where $a := 2g\sin\alpha$

b) length of infinitesimal path segment: $dl = dt\sqrt{\dot{x}^2+\dot{y}^2}$

1 Time needed to move across dl: $dT = dl/v(y)$

-> Passage time

$$T = \int_{t_-}^{t_+} dt\, \frac{1}{v(y)}\sqrt{\dot{x}^2+\dot{y}^2} = \int_{x_1}^{x_2} dx\, \frac{1}{v(y)}\sqrt{1+y'^2}$$

1 $$= \int_{x_1}^{x_2} dx\, L(y,y') \quad \text{with} \quad L(y,y') = \frac{1}{v(y)}\sqrt{1+y'^2}$$

cf. §2.3 remark (3)(ii) -> Jacobi's integral is a constant of motion, i.e.,

$$H(y,y') = y'\frac{\partial L}{\partial y'} - L = \frac{y'^2}{v(y)\sqrt{1+y'^2}} - \frac{1}{v(y)}\sqrt{1+y'^2} = \frac{-1}{v(y)\sqrt{1+y'^2}} = \text{const}$$

1 -> $y(1+y'^2) = \text{const} =: c_1 < 0$

c) substitute $y' = \cot t$, $t = \operatorname{arccot} y'$

(1) $\rightsquigarrow \underline{y} = \dfrac{c_1}{1+y'^2} = \dfrac{c_1}{1+\cot^2 t} = c_1 \sin^2 t = \underline{\tfrac{1}{2} c_1 (1-\cos 2t)}$

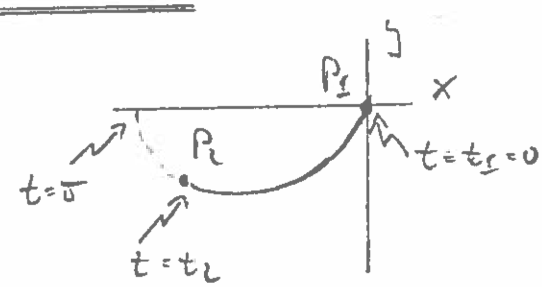
$dx = \dfrac{dy}{y'} = \dfrac{2c_1 \sin t \cos t \, dt}{\cot t} = 2c_1 \sin^2 t \, dt = c_1 (1-\cos 2t)\, dt$

(1) $\rightsquigarrow \underline{x} = c_2 + c_1 t - \tfrac{1}{2} c_1 \sin 2t = \underline{\tfrac{1}{2} c_1 (2t - \sin 2t) + c_2}$

initial conditions: $y_1 = y(t=0) = 0$ ✓

$x_1 = x(t=0) = c_2 \overset{!}{=} 0 \rightsquigarrow \underline{c_2 = 0}$

(1) let $c = \tfrac{1}{2} c_1 \rightsquigarrow$ $\boxed{\begin{aligned} x(t) &= c\,(2t - \sin 2t) \\ y(t) &= c\,(1 - \cos 2t) \end{aligned}}$ Brachistochrone in parametric form



remark: (1) This is for $x_2 < 0$

If $x_2 > 0$, consider $t < 0$

(2) c is a scale factor that must be chosen such that P_2 lies on the curve.

To bring this in standard form, let $\tau = 2t - \pi$, $\hat{x} = -x + \pi c$

$\rightsquigarrow \underline{\hat{x}(\tau)} = -c\left(\tau + \cancel{\pi} - \sin(\tau+\pi)\right) + \cancel{\pi} c = \underline{-c\,(\tau + \sin \tau)}$

(1) $\underline{y(\tau)} = c\left(1 - \cos(\tau+\pi)\right) = c\,(1+\cos\tau) = \underline{-c\,(-1-\cos\tau)}$ $\Big\}$ cycloid

d) Part c) =>

$$\boxed{\begin{array}{l} x(t) = c(2t - \sin 2t) \\ y(t) = c(1-\cos 2t) \end{array}}$$

$t = \operatorname{arcctg} y'$

$y' = dy/dx$

$c < 0$

$\rightsquigarrow \dot{x}(t) = 2c(1-\cos 2t)$

$\dot{y}(t) = 2c \sin 2t$

$\rightsquigarrow \dot{x}^2 + \dot{y}^2 = 4c^2(1-\cos 2t)^2 + 4c^2 \sin^2 2t = 4c^2 - 8c^2\cos 2t + 4c^2 = 8c^2(1-\cos 2t)$

1 $\qquad = 16c^2 \sin^2 t$

$\rightsquigarrow$ passage time for the brachistochrone

$$T_b = \int_0^{t_2} dt \, \frac{1}{\sqrt{-ay}} (-4c) \sin t = \frac{-4c}{\sqrt{-ac}} \int_0^{t_2} dt \, \frac{1}{\sqrt{1-\cos 2t}} \sin t$$

$$= \frac{-4c}{\sqrt{-ac}} \int_0^{t_2} dt \, \frac{\sin t}{\sqrt{2} \sin t} = \frac{-\sqrt{2} c}{\sqrt{-ac}} \, 2 \int_0^{t_2} dt$$

1 $\qquad = 2\sqrt{-2c/a} \; t_2$ with $t_2 = \operatorname{arcctg} y'_2$

e) straight line: $x(\tau) = x_2 \tau$, $y(\tau) = y_2 \tau$, $\tau_1 = 0$, $\tau_2 = 1$ $(y_2 < 0)$

$\rightsquigarrow$ passage time for straight line

1 $$T_s = \int_0^1 d\tau \, \frac{1}{\sqrt{-a y_2 \tau}} \sqrt{x_2^2 + y_2^2} = \sqrt{x_2^2+y_2^2} \, \frac{1}{\sqrt{-a y_2}} \int_0^1 d\tau \, \tau^{-1/2} = \frac{2}{\sqrt{-a y_2}} \sqrt{x_2^2 + y_2^2}$$

But we know that the point (x_2, y_2) lies on the brachistochrone

1 c) $\rightsquigarrow$ $x_2 = c(2t_2 - \sin 2t_2)$, $y_2 = c(1-\cos 2t_2)$

$\rightsquigarrow x_2^2 + y_2^2 = c^2(4t_2^2 - 4t_2 \sin 2t_2 + \sin^2 2t_2 + 1 - 2\cos 2t_2 + \cos^2 2t_2)$

$= c^2(2(1-\cos 2t_2) + 4t_2^2 - 4t_2 \sin 2t_2)$

$= c^2(4\sin^2 t_2 + 4t_2^2 - 4t_2 \sin 2t_2)$

(1) $= 4c^2\left(\sin^2 t_2 + t_2^2 - t_2 \sin 2t_2\right)$

$\rightsquigarrow T_S = \frac{2}{\sqrt{-ac}} \frac{1}{\sqrt{1-\cos 2t_2}} (-2c) \sqrt{\sin^2 t_2 + t_2^2 - t_2 \sin 2t_2}$

$= \frac{+2 \cdot \cancel{2} c}{\sqrt{-ac}} \frac{1}{\sqrt{\cancel{2}}\sin t_2} \sqrt{\sin^2 t_2 + t_2^2 - t_2 \sin 2t_2}$

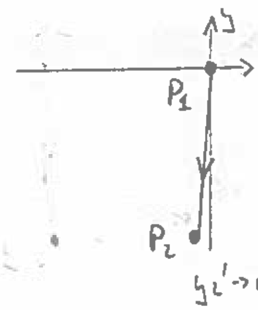
(1) $= \sqrt{-2c/a}\, \frac{+2}{\sin t_2} \sqrt{t_2^2 + \sin^2 t_2 - t_2 \sin 2t_2}$ with t_2 from d)

f) $t_2 \to 0$ ($\rightsquigarrow y_2' \to \infty$)

$T_s(t_2 \to 0) = \sqrt{-2c/a}\, \frac{-2}{t_2(1+O(t_2^2))} \sqrt{\cancel{t_2^2} + \cancel{t_2^2} - \frac{2}{6} t_2^4 - \cancel{2t_2^2} + \frac{8}{6} t_2^4 + O(t_2^6)}$

$= -2\sqrt{-2c/a}\; t_2 \left[1 + O(t_2^2)\right]$

(1) $= T_b \left[1 + O(t_2^2)\right]$ This is the case when the end point P_2 lies close to the fall line.



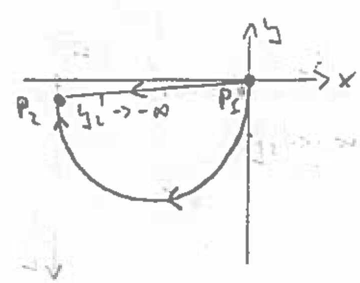
$\rightsquigarrow y_2' \to \infty$, $t_2 \to 0$, and brachistochrone

(1) and straight line are almost identical.

$t_2 = -\pi - \varepsilon$ ($\rightsquigarrow y_2' \to -\infty$)

(1) $T_S(t_2 = -\pi - \varepsilon) = \sqrt{-2c/a}\, \frac{2\pi}{\varepsilon} \left[1 + O(\varepsilon^2)\right] = T_b(t_2 = -\pi)\, \frac{1}{\varepsilon} \left[1 + O(\varepsilon^2)\right]$

This is the case when P_2 lies close to the x-axis $\rightsquigarrow y_2' \to -\infty$, and the straight-line path is very slow.



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