II.4.5. The Lindhard function revisited

Consider the function

$$f(t) = \frac{\sin t}{t}$$

Note that f is not absolutely integrable.

a) Find the Fourier transform $\hat{f}(\omega)$ by means of the techniques of ch. II § 3.4 (see in particular Problem II.3.2 for examples).

b) Show that $f(t)$ is a generalized function, therefore has a Fourier transform, and the Fourier inversion theorem holds. Show that $\hat{f}(\omega)$ from part a) is a generalized function. Perform a Fourier back transform and show that the result is $f(t)$, so the generalized-function approach is consistent with the approach in part a).

c) Find the Laplace transform $F(z)$ of $f(t)$ as defined in Problem II.4.3 and compare with the function analyzed in Problem II.2.1.

(6 points)

d) For extra credit, give an analogous analysis of the function $G(z)$ from Problem II.2.2 and the function $g(t)$ whose Laplace transform is $G(z)$.

Solution

II.4.5.) $f(t) = \frac{1}{t}\sin t$

a) $\hat{f}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \frac{\sin t}{t} = \int_{-\infty}^{\infty} dt\, \frac{1}{t} \sin t \cos(\omega t) + \underbrace{\int_{-\infty}^{\infty} dt\, \frac{1}{t} \sin t \sin(\omega t)}_{=0 \text{ in a CPV sense}}$

1

$$= \int_{-\infty}^{\infty} dt\, \frac{1}{t}\, \frac{1}{2i}\left(e^{it} - e^{-it}\right) \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$= \frac{1}{4i} \int_{-\infty}^{\infty} dt\, \frac{1}{t} \left(e^{i(1+\omega)t} + e^{i(1-\omega)t} - e^{-i(1-\omega)t} - e^{-i(1+\omega)t}\right)$$

$$= \frac{1}{4i} \Big\{ \mathrm{sgn}(\omega+1) \Big[\int_{-\infty}^{\infty} dt\, \frac{1}{t} e^{it} - \int_{-\infty}^{\infty} dt\, \frac{1}{t} e^{-it}\Big] + \mathrm{sgn}(\omega-1) \Big[\int_{-\infty}^{\infty} dt\, \frac{1}{t} e^{-it} - \int_{-\infty}^{\infty} dt\, \frac{1}{t} e^{it}\Big] \Big\}$$

$$= \frac{1}{4i} \Big[\mathrm{sgn}(\omega+1) \int_{-\infty}^{\infty} dt\, \frac{1}{t}\left(e^{it} - e^{-it}\right) - \mathrm{sgn}(\omega-1) \int_{-\infty}^{\infty} dt\, \frac{1}{t}\left(e^{it} - e^{-it}\right) \Big]$$

$$= \frac{1}{4i} \left[\mathrm{sgn}(\omega+1) - \mathrm{sgn}(\omega-1)\right] \int_{-\infty}^{\infty} dt\, \frac{1}{t}\left(e^{it} - e^{-it}\right)$$

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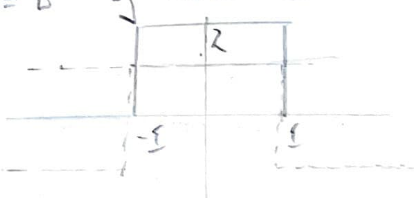
$$= \frac{1}{2} \left[\mathrm{sgn}(\omega+1) - \mathrm{sgn}(\omega-1)\right] \underbrace{\int_{-\infty}^{\infty} dt\, \frac{\sin t}{t}}_{=\pi \text{ by Problem II.3.2}}$$



$$= \frac{\pi}{2} \left[\mathrm{sgn}(\omega+1) - \mathrm{sgn}(\omega-1)\right]$$

$$= \pi\, \Theta(1-\omega^2)$$

1

b) $f(t)$ is a generalized fct. by ch II §4.4 Theorem 3.

ch II §4.4. Prop. 7 $\rightarrow$ Its FT $\hat{f}(\omega)$ exists and is itself a generalized fct.

$\rightarrow$ the Fourier inversion theorem holds, i.e., $f(t)$ is the FBT of $\hat{f}(\omega)$.

(1)

Now take $\hat{f}(\omega)$ from part a) and Fourier back transform:

$$\underline{f(t)} = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega) = \int_{-1}^{1} \frac{d\omega}{2\pi} e^{-i\omega t}\, \pi = \frac{1}{2} \int_{-1}^{1} d\omega\, e^{-i\omega t}$$

$$= \frac{1}{2} \frac{1}{-it} e^{-i\omega t}\Big|_{-1}^{1} = -\frac{1}{t} \frac{1}{2i} \left(e^{-it} - e^{it}\right) = \underline{\underline{\frac{\sin t}{t}}}$$

$\rightarrow$ Everything is consistent, i.e., <u>the Cauchy Principal Value analysis in part a) agrees with the generalized function analysis!</u>

(1)

c) 1st case: $\operatorname{Im} z > 0$

$$F(z) = \int_0^\infty dt\, e^{izt} \frac{\sin t}{t} = i \operatorname{arctanh} \frac{1}{z} = \frac{i}{2} \log \frac{1+1/z}{1-1/z} = \frac{i}{2} \log \frac{z+1}{z-1}$$

2nd case: $\operatorname{Im} z < 0$

$$F(z) = -\int_{-\infty}^{0} dt\, e^{izt} \frac{\sin t}{t} = -\int_0^\infty dt\, e^{-izt} \frac{\sin t}{t} = -\frac{i}{2} \log \frac{-z+1}{-z-1} = \frac{i}{2} \log \frac{z+1}{z-1}$$

(1) $\rightarrow$ $\underline{\underline{F(z) = \frac{i}{2} \log \frac{z+1}{z-1}}}$ $\rightarrow$ the fct. $F(z)$ from II.2.1 is $-\frac{\pi}{2}$ times the Laplace trafo of $\frac{\sin t}{t}$!