II.4.4. Generalized functions derived from generalized functions

Prove Proposition 7 in ch.2 §4.4, which says

Proposition Let $f(x)$ and $g(x)$ be generalized functions defined by sequences $f_n(x)$ and $g_n(x)$. Then the following are all generalized functions:

a) $f(x) + g(x)$ defined by the sequence $f_n(x) + g_n(x)$
b) $f'(x)$ defined by $f'_n(x)$
c) $h(x) = f(ax + b)$ defined by the sequence $f_n(ax + b)$
d) $\varphi(x)\, f(x)$ defined by the sequence $\varphi(x)\, f_n(x)$ with φ a fairly good function,
e) $\hat{f}(k)$ defined by the sequence $\hat{f}_n(k) = FT(f_n)(k)$.

(7 points)

Solution

For each of the statements in the proposition we must show
(i) the sequence in question is a sequence of good functions,
(ii) the sequence in question is a regular sequence,
(iii) different choices of equivalent sequences f_n, g_n lead to equivalent sequences that define the new functions.

Property (i) is true for all statements by §4.3 remark (2). 1pt

Now check (ii) and (iii) for the various statements:

a) $\lim_{n\to\infty} \int dx\, (f_n(x) + g_n(x))\, F(x) = \lim_{n\to\infty} \int dx\, f_n(x)\, F(x) + \lim_{n\to\infty} \int dx\, g_n(x)\, F(x)$
The limits on the rhs exist since f_n and g_n are regular sequences, and hence the limit on the lhs exists, so (ii) is true. 1pt

Also, different equivalent sequences f_n and g_n lead to the same limiting values on the rhs $\Rightarrow$ the resulting sequences $f_n + g_n$ are all equivalent, so (iii) is true. 1pt

b) $\lim_{n\to\infty} \int dx\, f'_n(x)\, F(x) = -\lim_{n\to\infty} \int dx\, f_n(x)\, F'(x)$
But $F' \in \gamma \Rightarrow$ the limit on the rhs exists and is the same for all equivalent sequences $f_n(x)$.
$\Rightarrow$ The sequences $f'_n(x)$ on the lhs are regular and equivalent of the sequences f_n are equivalent.
$\Rightarrow$ (ii) and (ii) are true for Statement b). 1pt

The same arguments apply to

c) $\lim{}_n \to \infty \int dx\, f_n(ax + b)\, F(x) = \frac{1}{|a|} \lim_{n\to\infty} \int dx\, f_n(x)\, F((x - b)/2)$ 1pt

and

d) $\lim_{n\to\infty} \int dx\, (\varphi(x)\, f_n(x))\, F(x) = \lim_{n\to\infty} \int dx\, f_n(x)\, (\varphi(x)\, F(x))$ 1pt

since $F((x - b)/a)$ and $\varphi(x)\, F(x)$ are both good functions.

Finally,

e) By Parseval's theorem we have $\lim_{n\to\infty} \int dk\, \hat{f}_n(k)\, \hat{F}(k) = \lim_{n\to\infty} \int dx\, f_n(x)\, F(-x)$
and hence the same arguments apply again. 1pt