II.4.3. More 1-d Fourier transforms

Consider a function of time $f(t)$ and define its Fourier transform

$$\hat{f}(\omega) := \int dt\; e^{i\omega t}\, f(t)$$

and its Laplace transform $F(z)$ as

$$F(z) = \pm \int dt\, e^{izt}\, f_{\pm}(t) \qquad (\pm \text{ for } \operatorname{sgn}(\operatorname{Im} z) = \pm 1)$$

with z a complex frequency and $f_{\pm}(t) = \Theta(\pm t)\, f(t)$. Further define

$$F''(\omega) = \frac{1}{2i}\left[F(\omega + i0) - F(\omega - i0)\right] \qquad , \qquad F'(\omega) = \frac{1}{2}\left[F(\omega + i0) + F(\omega - i0)\right]$$

Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

hint: $\lim_{\epsilon\to 0} \epsilon/(x^2+\epsilon^2) = \pi\,\delta(x)$, with $\delta(x)$ the familiar Dirac delta-function, which we will study in detail in ch. II §4.5.

Show that in both cases $\int \frac{d\omega}{\pi}\,\frac{F''(\omega)}{\omega} = F'(\omega = 0)$.

note: These concepts are important for the theory of response functions.

(4 points)

Solution

We have

$$F(\omega \pm i0) = \pm \int dt\, \Theta(\pm t)\, e^{i(\omega \pm i0)t}\, f(t) = \pm \hat{f}_{\pm}(\omega)$$

(Note that the $\pm i0$ in the exponent ensures convergence!)

a) $\hat{f}_{-}(\omega) = \int_{-\infty}^{0} dt\, e^{i\omega t}\, e^{-t/\tau} = \frac{-1}{i\omega - 1/\tau} = \frac{\tau}{1 - i\omega\tau}$

$\hat{f}_{+}(\omega) = \int_{0}^{\infty} dt\, e^{i\omega t}\, e^{t/\tau} = \frac{1}{i\omega - +1/\tau} = \frac{\tau}{1 + i\omega\tau}$ 1pt

Therefore,

$F''(\omega) = \frac{1}{2i}\left(\hat{f}_{+}(\omega) + \hat{f}_{-}(\omega)\right) = \frac{1}{2i}\left(\frac{\tau}{1-i\omega\tau} + \frac{\tau}{1+i\omega\tau}\right) = \frac{2\tau}{2i}\,\frac{1}{1+(\omega\tau)^2} = \frac{-i\tau}{1+(\omega\tau)^2}$

$F'(\omega) = \frac{1}{2}\left(\hat{f}_{+}(\omega) - \hat{f}_{-}(\omega)\right) = \frac{1}{2}\left(\frac{\tau}{1-i\omega\tau} - \frac{\tau}{1+i\omega\tau}\right) = \frac{i\omega\tau^2}{1+(\omega\tau)^2}$

Therefore,

$\int \frac{d\omega}{\pi}\,\frac{F''(\omega)}{\omega} = 0 = F'(\omega = 0)$ 1pt

b) $F(\omega+i0)=\int_0^\infty dt\, e^{i(\omega+i0)t}\, e^{i\omega_0 t}=\frac{-1}{i\omega-0+i\omega_0}=\frac{i}{\omega+\omega_0+i0}$
$F(\omega-i0)=-\int_{-\infty}^0 dt\, e^{i(\omega-i0)t}\, e^{i\omega_0 t}=\frac{-1}{i\omega+0+i\omega_0}=\frac{i}{\omega+\omega_0-i0}$

Therefore

$F''(\omega)=\frac{i}{2i}\left(\frac{1}{\omega+\omega_0+i0}-\frac{1}{\omega+\omega_0-i0}\right)=\frac{1}{2}\,\frac{-2i0}{(\omega+\omega_0)^2+0^2}=-i\pi\,\delta(\omega+\omega_0)$
$F'(\omega)=\frac{i}{2}\left(\frac{1}{\omega+\omega_0+i0}+\frac{1}{\omega+\omega_0-i0}\right)=i\,\frac{\omega+\omega_0}{(\omega+\omega_0)^2+0^2}=\frac{i}{\omega+\omega_0}$ 1pt

Therefore,

$\int \frac{d\omega}{\pi}\,\frac{F''(\omega)}{\omega}=\frac{-i}{-\omega_0}=\frac{i}{\omega_0}=F'(\omega=0)$ 1pt