II.4.1. 1-d Fourier transforms

Consider a function f of one real variable x. Calculate the Fourier transforms $\hat{f}(k) = \int dx\, e^{-ikx}\, f(x)$ of the following functions:

a) $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \quad .$

b) $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \quad .$

c) $f(x) = e^{-(x/x_0)^2} \quad .$

(3 points)

Solution

a) $\hat{f}(k) = \int dx\, e^{-ikx} \Theta(|x| \le 1) = \int_{-1}^{1} dx\, e^{-ikx} = \frac{-1}{ik}\left(e^{-ikx} - e^{ikx}\right) = \frac{1}{ik}\, 2i\, \sin k = \frac{2}{k}\, \sin k$ 1pt

b)

$$\begin{aligned}
\hat{f}(k) &= \int_{-1}^{1} dx\, e^{-ikx}\,(1 - |x|) = \int_{-1}^{0} dx\, e^{-ikx} (1 + x) + \int_{0}^{1} dx\, e^{-ikx} (1 - x) \\
&= \frac{2}{k}\, \sin k + \frac{e^{-ikx}}{-ik} \left(x - \frac{1}{-ik} \right)\Bigg|_{-1}^{0} - \frac{e^{-ikx}}{-ik} \left(x - \frac{1}{-ik} \right)\Bigg|_{0}^{1} \\
&= \frac{2}{k}\, \sin k - \frac{1}{(ik)^2} + \frac{1}{ik}\, e^{ik} \left(-1 + \frac{1}{ik} \right) + \frac{1}{ik}\, e^{-ik} \left(1 + \frac{1}{ik} \right) - \frac{1}{(ik)^2} \\
&= \frac{2}{k}\, \sin k + \frac{2}{k^2} - \frac{1}{ik}\, 2i\, \sin k + \frac{1}{(ik)^2}\, 2 \cos k \\
&= \frac{2}{k^2} (1 - \cos k)
\end{aligned}$$

1pt

c)

$$\begin{aligned}
\hat{f}(k) &= \int d\, e^{-ikx}\, e^{-(x/x_0)^2} = x_0 \int dx\, e^{-x^2 - ikx_0 x} \\
&= x_0 \int dx\, e^{-(x^2 + 2i(kx_0/2)x - k^2 x_0^2/4)} e^{-k^2 x_0^2/4} \\
&= x_0\, e^{-k^2 x_0^2/4} \int dx\, e^{-(x + ikx_0/2)^2} \\
&= x_0\, e^{-k^2 x_0^2/4} \int dx\, e^{-x^2} \\
&= \sqrt{\pi}\, x_0\, e^{-k^2 x_0^2/4}
\end{aligned}$$

1pt