II.3.2. **Applications of the residue theorem**

Use complex analysis to evaluate the real integrals

a)

$$\int_{-\infty}^{\infty} dx \, \frac{1}{x^4+1}$$

b)

$$\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x}$$

hint: Write $\sin x = (e^{ix} - e^{-ix})/2i$ and consider the resulting two integrals with complex integrands. Why is this a good strategy?

c)

$$\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} \, \frac{1}{1+x^2}$$

and check your results by means of Wolfram Alpha.

Let $a \in \mathbb{C}$ with $\mathrm{Re}\, a > 0$. Use the residue theorem to show that

d)

$$\int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\pi/a}$$

Now let $a \in \mathbb{R}$ and consider the integral

e)

$$\int_{-\infty}^{\infty} \frac{dx}{x} \, \frac{1}{x^2+a^2}$$

and define its Cauchy principal value by

$$\lim_{R\to 0} [\int_{-\infty}^{-R} dx \, f(x) + \int_{R}^{\infty} dx \, f(x)]$$

with $f(x) = 1/x(x^2+a^2)$. Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

hint: Go around the pole on a semicircle of radius R and let $R \to 0$.

(17 points)

Solution

2.3.2.) a) $I = \int_{-\infty}^{\infty} dx \, \frac{1}{x^4+1}$

Consider $f(z) = \frac{1}{z^4+1} = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

poles: $z^4 = -1 \rightsquigarrow z_1 = e^{i\pi/4}, \; z_2 = e^{3i\pi/4}$

$z_3 = e^{5i\pi/4}, \; z_4 = e^{7i\pi/4}$

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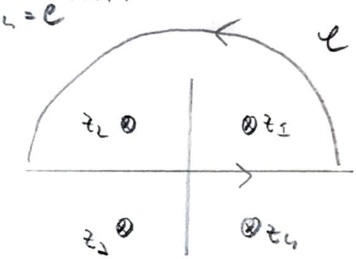

$\rightsquigarrow \underline{\underline{I}} = \int_{\mathcal{C}} dz \, f(z)$

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$= \underline{2\pi i \left[\operatorname{Res} f(z_1) + \operatorname{Res} f(z_2)\right]}$

$\underline{\operatorname{Res} f(z_1)} = (z-z_1) f(z)\Big|_{z=z_1} = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$

$= \frac{1}{(e^{i\frac{\pi}{4}} - e^{i\frac{3\pi}{4}})(e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}})(e^{i\frac{\pi}{4}} - e^{i\frac{7\pi}{4}})} = \frac{e^{-i\pi/4}}{(1-i)(1+1)(1+i)}$

$= \underline{\frac{1}{4} e^{-i\pi/4}}$

$\underline{\operatorname{Res} f(z_2)} = \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{e^{-i3\pi/4}}{(1+i)(1-i)(1+1)} = \underline{\frac{1}{4} e^{-i3\pi/4}}$

$\rightsquigarrow \underline{\underline{I}} = 2\pi i \frac{1}{4} e^{-i\pi/4} \left(1 + e^{-i\pi/2}\right) = \frac{i\pi}{2} \frac{1}{\sqrt{2}} (1-i)^2$

$= \frac{i\pi}{2\sqrt{2}} (-2i) = \underline{\underline{\frac{\pi}{\sqrt{2}}}}$

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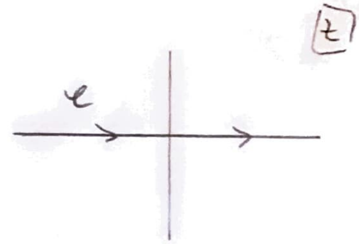


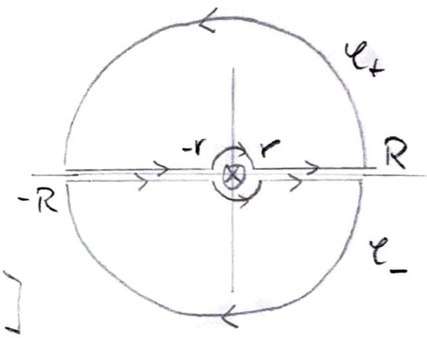
b) $\underline{\underline{\mathcal{I}}} = \int dx \frac{\sin x}{x} = \int_{\mathcal{C}} dt \frac{\sin t}{t}$

$= \frac{1}{2i} \int_{\mathcal{C}} dt \frac{1}{t} \left[e^{it} - e^{-it} \right]$

remark: $\frac{\sin t}{t}$ is analytic everywhere

$\frac{1}{t} e^{\pm it}$ has a simple pole at $t=0$ with residue 1.

consider



$\tilde{\mathcal{J}}_\pm(R,r) := \frac{1}{2i} \int_{\mathcal{C}_\pm} dt \frac{1}{t} e^{\pm it}$

$= \frac{1}{2i} \left[\int_{-R}^{r} dx \frac{1}{x} e^{\pm ix} + \int_{r}^{R} dx \frac{1}{x} e^{\pm ix} \right]$

$+ \frac{1}{2i} \int_{c_\pm} dt \frac{1}{t} e^{\pm it} + \frac{1}{2i} \int_{C_\pm} dt \frac{1}{t} e^{\pm it}$

(1) $=: \mathcal{J}_\pm(R,r) + \frac{1}{2i} \int_{c_\pm} dt \frac{1}{t} e^{\pm it} + \frac{1}{2i} \int_{C_\pm} dt \frac{1}{t} e^{\pm it}$

where $\mathcal{J}_\pm(R,r) = \int_{-R}^{-r} dx \frac{1}{x} e^{\pm ix} + \int_{r}^{R} dx \frac{1}{x} e^{\pm ix}$

and $c_\pm$ = semicircle with radius r

$C_\pm$ = semicircle with radius R

$\rightsquigarrow \underline{\underline{\mathcal{I}}} = \lim_{\substack{r \to 0 \\ R \to \infty}} \left[\mathcal{J}_+(R,r) - \mathcal{J}_-(R,r) \right]$

(1)

Parametrization
$$C_+: t = re^{i\varphi} \quad (\pi \leq \varphi \leq 0)$$
$$C_-: t = re^{i\varphi} \quad (-\pi \leq \varphi \leq 0)$$
$$dt = ire^{i\varphi}\,d\varphi$$
$$C'_+: t = Re^{i\varphi} \quad (0 \leq \varphi \leq \pi)$$
$$C'_-: t = Re^{i\varphi} \quad (0 \geq \varphi \geq -\pi)$$
$$dt = iRe^{i\varphi}\,d\varphi$$

$$\rightsquigarrow \underline{\int_{C_+} dt\,\frac{1}{t}e^{it}} = ir\int_\pi^0 d\varphi\,\cancel{e^{i\varphi}}\,\frac{1}{r\cancel{e^{i\varphi}}}\,e^{ire^{i\varphi}} = -i\int_0^\pi d\varphi\,e^{ire^{i\varphi}}$$

$$\overset{r\to 0}{=} \underline{-i\pi + O(r)}$$

$$\underline{\int_{C_-} dt\,\frac{1}{t}e^{-it}} = ir\int_{-\pi}^0 d\varphi\,\cancel{e^{i\varphi}}\,\frac{1}{r\cancel{e^{i\varphi}}}\,e^{-ire^{i\varphi}} = \underline{i\pi + O(r)}$$

1

$$\underline{\int_{C'_+} dt\,\frac{1}{t}e^{it}} = i\cancel{R}\int_0^\pi d\varphi\,\cancel{e^{i\varphi}}\,\frac{1}{\cancel{Re^{i\varphi}}}\,e^{iR(\cos\varphi + i\sin\varphi)} = i\int_0^\pi d\varphi\,e^{iR\cos\varphi - R\sin\varphi}$$

$$\underline{\int_{C'_-} dt\,\frac{1}{t}e^{-it}} = -iR\int_{-\pi}^0 d\varphi\,\cancel{e^{i\varphi}}\,\frac{1}{\cancel{Re^{i\varphi}}}\,e^{iR\cos\varphi + R\sin\varphi} = i\int_0^\pi d\varphi\,e^{iR\cos\varphi - R\sin\varphi}$$

$$\rightsquigarrow \underline{\left|\int_{C'_\pm} dt\,\frac{1}{t}e^{\pm it}\right|} = \left|\int_0^\pi d\varphi\,e^{iR\cos\varphi - R\sin\varphi}\right| \leq \int_0^\pi d\varphi\,\underbrace{\left|e^{iR\cos\varphi}\right|}_{=1}\cdot e^{-R\sin\varphi}$$

$$= 2\int_0^{\pi/2} d\varphi\,e^{-R\sin\varphi} = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}}\,e^{-Rx}$$

$$= \frac{2}{R}\int_0^R dx\,\frac{1}{\sqrt{1-x^2/R^2}}\,e^{-x} = \frac{2}{R}\left[\int_0^\infty dx\,e^{-x} + O(e^{-R})\right]$$

$$= \underline{\frac{2}{R}\left[1 + O(e^{-R})\right] \to 0 \quad \text{for } R\to\infty}$$

1

(This also follows from Jordan's Lemma, \$3.4 Lemma 5)

Now, $\tilde{\gamma}_\pm(R,r) = 0$ by the residue theorem

$$\rightsquigarrow \quad \underline{\underline{I}} = \lim_{\substack{r \to 0 \\ R \to \infty}} \left[J_+(R,r) - J_-(R,r) \right]$$

$$= \lim_{\substack{r \to 0 \\ R \to \infty}} \Big[\tilde{J}_+(R,r) - \frac{1}{2i} \int_{c_+} dt \, \frac{1}{t} e^{it} - \frac{1}{2i} \int_{C_+} dt \, \frac{1}{t} e^{it}$$

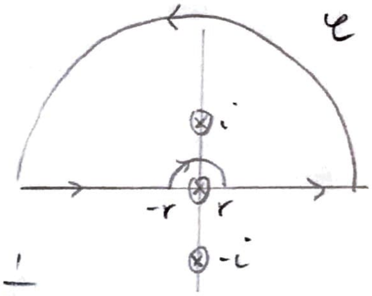
$$- \tilde{J}_-(R,r) + \frac{1}{2i} \int_{c_-} dt \, \frac{1}{t} e^{-it} + \frac{1}{2i} \int_{C_-} dt \, \frac{1}{t} e^{-it} \Big]$$

1 $$= 0 - \frac{1}{2i}(-i\pi) - 0 - 0 + \frac{1}{2i} i\pi + 0 = \frac{\pi}{2} + \frac{\pi}{2} = \underline{\underline{\pi}}$$

c) $\underline{\underline{J}} = \int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} \frac{1}{x^2+1} = \int_{-\infty}^{\infty} dx \, \frac{1}{x} \frac{1}{\cancel{2}i} (e^{ix} - e^{-ix}) \frac{1}{x^2+1}$

$$= -i \int_{-\infty}^{\infty} dx \, \underbrace{\frac{e^{ix}}{x} \frac{1}{x^2+1}} = -i \int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x} \frac{1}{(x+i)(x-i)}$$

$$= -i \int_{-\infty}^{\infty} dx \, f(x)$$



(1) $f(|z| \to 0)$ for $\operatorname{Im} z \geq 0$

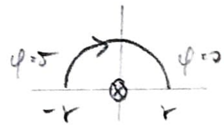
$$\rightsquigarrow \int_{\mathcal{C}} dz \, f(z) = 2\pi i \operatorname{Res} f(z=i) = 2\pi \cancel{i} \frac{e^{-1}}{i} \frac{1}{2\cancel{i}} = -i\pi \frac{1}{e}$$

$$\| $$

$$\int_{-\infty}^{\infty} dx \, f(x) + \lim_{r \to 0} \int_{\frown} dz \, f(z)$$

$$\rightsquigarrow \int_{-\infty}^{\infty} dx \, f(x) = -\frac{i\pi}{e} - \lim_{r \to 0} \int_{\frown} dz \, f(z)$$

(1)



Along the semicircle, $z = r(\cos\varphi + i\sin\varphi)$ with $r =$ const.

$$\rightsquigarrow dz = r(-\sin\varphi + i\cos\varphi)\, d\varphi = ir(\cos\varphi + i\sin\varphi)\, d\varphi$$

$$\rightsquigarrow \int_{\frown} dz \, f(z) = i\cancel{r} \int_{\pi}^{0} d\varphi \, \cancel{(\cos\varphi + i\sin\varphi)} \frac{e^{ir(\cos\varphi + i\sin\varphi)}}{\cancel{r(\cos\varphi + i\sin\varphi)}} \left[1 + O(r^2)\right]$$

$$= -i\pi + O(r)$$

$$\rightsquigarrow \int_{-\infty}^{\infty} dx \, f(x) = -\frac{i\pi}{e} + i\pi = \frac{i\pi}{e}(e-1)$$

(1) $\rightsquigarrow \underline{\underline{J = \frac{\pi}{e}(e-1)}}$

p 2.3.2-6

WolframAlpha computational... knowledge engine

integral

≡ Examples ⤮ Random

Assuming "integral" refers to a computation | Use as a general topic or a character or referring to a mathematical definition or a word instead

- function to integrate: sin x/(x(1+x^2))
- lower limit: -Infinity
- upper limit: Infinity

Also include: variable

Definite integral:

More digits

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x\left(1+x^2\right)}\, dx = \frac{(e-1)\,\pi}{e} \approx 1.98587$$

Computed by Wolfram *Mathematica*

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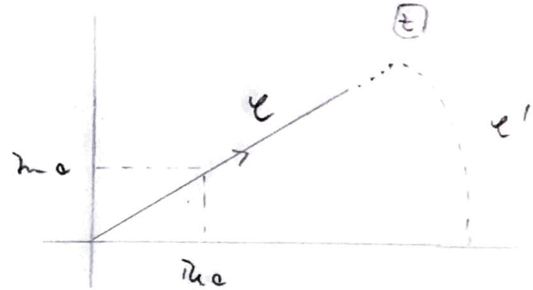
d) $I := \int_{-\infty}^{\infty} dx\, e^{-ax^2} = 2\int_0^{\infty} dx\, e^{-ax^2}$ $\qquad \underline{\mathrm{Re}\, a > 0}$

$z = \sqrt{a}\,x \;, \quad dz = \sqrt{a}\, dx$



(1) $\to \underline{I = \frac{2}{\sqrt{a}} \int_{C} dz\, e^{-z^2}}$

On the other hand, the residue theorem yields

(1) $\int_{C} dz\, e^{-z^2} + \int_{C'} dz\, e^{-z^2} + \int_{\infty}^{0} dz\, e^{-z^2} = 0$

$\to \underline{\frac{1}{2}\sqrt{a}\, I} = \int_{C} dz\, e^{-z^2} = -\underbrace{\int_{C'} dz\, e^{-z^2}}_{=0} + \int_{0}^{\infty} dz\, e^{-z^2} = \int_{0}^{\infty} dx\, e^{-x^2}$

$= \frac{1}{2} \int_{-\infty}^{\infty} dx\, e^{-x^2} = \frac{1}{2} \left[\int_{-\infty}^{\infty} dx\, dy\, e^{-(x^2+y^2)} \right]^{1/2}$

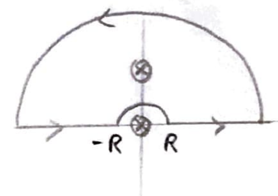
$= \frac{1}{2} \left(\int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r\, e^{-r^2} \right)^{1/2} = \frac{1}{2} \Big(2\pi\, \frac{1}{2} \underbrace{\int_0^{\infty} dx\, e^{-x}}_{=1} \Big)^{1/2} = \underline{\frac{\sqrt{\pi}}{2}}$

(1) $\to \underline{\underline{I = \sqrt{\pi/a}}}$

c) $\underline{\mathcal{I}} = \int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{a^2+x^2} = \underline{\frac{1}{a^2}\mathcal{J}}$,

$\mathcal{J} = \int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{1+x^2} = \int_{-\infty}^{\infty} \frac{dz}{z} \frac{1}{1+z^2}$

remark: This integral exists only in the sense of a principal value



(1) $\underline{\mathcal{J}} = \lim_{R\to 0} \int_{C} \frac{dz}{z}\frac{1}{1+z^2} - \lim_{R\to 0} \int_{m} \frac{dz}{z}\frac{1}{1+z^2}$

poles: $z=0$, $z=\pm i$

$\mathrm{Res}\left(\frac{1}{z}\frac{1}{1+z^2}, z=i\right) = \frac{1}{i}\frac{1}{2i} = \frac{-1}{2}$

$\rightsquigarrow \underline{\mathcal{J}} = 2\pi i \, \frac{-1}{2} - \lim_{R\to 0} \int_{\pi}^{0} d\varphi \, iR \frac{e^{i\varphi}}{Re^{i\varphi}} \frac{1}{1+R^2 e^{2i\varphi}} = -\pi i - i\int_{\pi}^{0} d\varphi$

$= -\pi i - i(-\pi) = \underline{0}$

(1) $\rightsquigarrow \underline{\mathcal{I} = 0}$

remark: This result is consistent with the usual principal value for an antisymmetric integral:

$\underline{\mathrm{P.V.}(\mathcal{I})} = \lim_{R\to 0}\left[\int_{-\infty}^{-R} \frac{dx}{x}\frac{1}{a^2+x^2} + \int_{R}^{\infty} \frac{dx}{x}\frac{1}{a^2+x^2}\right] =$

(1) $= \lim_{R\to 0} \int_{R}^{\infty} dx \left[\frac{1}{x}\frac{1}{a^2+x^2} - \frac{1}{x}\frac{1}{a^2+x^2}\right] = \underline{0}$