II.3.1. Laurent series

Find the Laurent series for the function

$$f(z) = 1/(z^2+1)$$

in the point $z = i$. That is, find the coefficients f_n that enter the theorem in ch. 2 §3.2.

(3 points)

Solution:

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{z-i}\,\frac{1}{2i+(z-i)} = \frac{1}{z-i}\,\frac{1}{2i}\,\frac{1}{1+\frac{1}{2i}(z-i)}$$

1 pt

Now expand

$$\frac{1}{1+\frac{1}{2i}(z-i)} = \sum_{n=0}^{\infty}(-)^n\,\frac{(z-i)^n}{(2i)^n}$$

$$\Rightarrow \quad f(z) = \frac{1}{z-i}\,\frac{1}{2i}\sum_{n=0}^{\infty}(-)^n\,\frac{(z-i)^n}{(2i)^n} = \sum_{n=0}^{\infty}(-)^n\frac{(z-i)^{n-1}}{(2i)^{n+1}} = \sum_{n=-1}^{\infty}\frac{(-)^{n+1}}{(2i)^{n+2}}\,(z-i)^n$$

1 pt

$\Rightarrow$ The coefficients f_n in the Laurent series are

$$f_n = \begin{cases} (-)^{n+1}/(2i)^{n+2} & \text{for} \qquad n \geq -1 \\ 0 & \text{for} \qquad n < -1 \end{cases}$$

1 pt