II.2.4. **Exponentials**

Consider the exponential function

$$f(z) = e^z = e^{z'+iz''}$$

a) Show that $f(z)$ is analytic everywhere in $\mathbb{C}$.

b) Convince yourself explicitly that the real and imaginary parts of f obey Laplace's differential equation.

c) Show that $df/dz|_z = f(z)$.

d) Show that $\cos z$ and $\sin z$, defined by

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \quad , \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

are analytic everywhere in $\mathbb{C}$, and that

$$\frac{d}{dz}\cos z = -\sin z \quad , \quad \frac{d}{dz}\sin z = \cos z \ .$$

(4 points)

Solution

2.2.4. a) $f(z) = e^{z} = e^{z'+iz''} = e^{z'}e^{iz''} = e^{z'}(\cos z'' + i\sin z'')$

$$\to \quad \underline{f'(z',z'') = e^{z'}\cos z''}$$

$$\underline{f''(z',z'') = e^{z'}\sin z''}$$

$$\to \quad \frac{\partial f'}{\partial z'} = e^{z'}\cos z'' = \frac{\partial f''}{\partial z''}$$

$$\frac{\partial f'}{\partial z''} = -e^{z'}\sin z'' = -\frac{\partial f''}{\partial z'}$$

1 $\to$ <u>f is analytic</u> <u>on $\mathbb{C}$</u>

b) $$\frac{\partial^2 f'}{\partial z'^2} + \frac{\partial^2 f'}{\partial z''^2} = e^{z'}\cos z'' - e^{z'}\cos z'' = 0$$

1 $$\frac{\partial^2 f''}{\partial z'^2} + \frac{\partial^2 f''}{\partial z''^2} = e^{z'}\sin z' - e^{z'}\sin z' = 0$$

c) Approach z along the real axis

$$\to \quad \frac{df}{dz}(z) = \frac{\partial f'}{\partial z'}\bigg|_z + i\frac{\partial f''}{\partial z'}\bigg|_z = e^{z'}\cos z'' + ie^{z'}\sin z'' = e^{z'}(\cos z'' + i\sin z'')$$

1 $$= e^{z'}e^{iz''} = e^{z} = \underline{f(z)} \qquad \square$$

d) Consider $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

e^{z} is analytic everywhere $\to$ so is $e^{\pm iz}$ $\to$ so are $\cos z$ and $\sin z$

$$\frac{d}{dz}\cos z = \frac{i}{2}(e^{iz} - e^{-iz}) = \frac{-1}{2i}(e^{iz} - e^{-iz}) = \underline{-\sin z}$$

$$\frac{d}{dz}\sin z = \frac{i}{2i}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \underline{\cos z}$$