II.2.3. **Proof of the Cauchy-Riemann Theorem**

Prove the Cauchy-Riemann theorem from ch.2 §2.2:

a) Let $f(z) = f'(z', z'') + if''(z', z'')$ be analytic everywhere in $\Omega \subseteq \mathbb{C}$. Show that the Cauchy-Riemann equations

$$\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''} \qquad \text{and} \qquad \frac{\partial f'}{\partial z''} = -\frac{\partial f''}{\partial z'}$$

hold $\forall z \in \Omega$.

hint: Start with the difference quotient $(f(z) - f(z_0))/(z - z_0)$ and require that its limit for $z \to z_0$ exists if z_0 is approached on paths either parallel to the real axis, or parallel to the imaginary axis.

b) Let the Cauchy-Riemann equations hold in a point $z_0 \in \Omega$. Show that this implies that f is analytic in the point z_0.

hint: Consider $f(z) - f(z_0)$ and expand $f'(z', z'')$ and $f''(z', z'')$ in Taylor series about z_0.

(8 points)

Solution

2.2.3 a) Consider the difference quotient

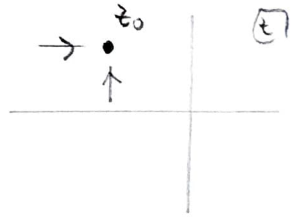
$$\frac{\Delta f}{\Delta z} = \frac{f^1(z^1,z^2) + i f^2(z^1,z^2) - f^1(z_0^1,z_0^2) - i f^2(z_0^1,z_0^2)}{z^1 + i z^2 - z_0^1 - i z_0^2}$$

for f in the vicinity of a point $z_0 \in \mathbb{R}$.

For the limit $\frac{df}{dz} = \lim_{z \to z_0} \frac{\Delta f}{\Delta z}$ to exist, it must exist for f^1 and f^2 separately, and it must exist for $z \to z_0$ along any path. In particular, it must exist for $z \to z_0$ along the real axis and along the imaginary axis.

1 → If f is differentiable in z_0, then the derivations $\partial f^1/\partial z^1$, $\partial f^1/\partial z^2$, $\partial f^2/\partial z^1$, $\partial f^2/\partial z^2$ all must exist in z_0

Now approach z_0 parallel to the real axis, i.e., for fixed z^2:



1 $$\left.\frac{df}{dz}\right|_{\substack{z_0 \\ z^2 = \text{const}}} = \left.\frac{\partial f^1}{\partial z^1}\right|_{z_0} + i \left.\frac{\partial f^2}{\partial z^1}\right|_{z_0} \qquad (*)$$

Now approach for fixed z^1:

1 $$\left.\frac{df}{dz}\right|_{\substack{z_0 \\ z^1 = \text{const}}} = \left.\frac{\partial f^1}{i\,\partial z^2}\right|_{z_0} + i \left.\frac{\partial f^2}{i\,\partial z^2}\right|_{z_0} = \left.\frac{\partial f^2}{\partial z^2}\right|_{z_0} - i \left.\frac{\partial f^1}{\partial z^2}\right|_{z_0} \qquad (**)$$

But if f is differentiable in z_0, then $(*) = (**)$

1 → $\boxed{\frac{\partial f^1}{\partial z^1} = \frac{\partial f^2}{\partial z^2}}$ and $\boxed{\frac{\partial f^2}{\partial z^1} = -\frac{\partial f^1}{\partial z^2}}$ → The Cauchy-Riemann eqs are necessary

b) Consider the numerator of the difference quotient,

$$f(z)-f(z_0) = f'(z',z'') + i f''(z',z'') - f'(z_0',z_0'') - i f''(z_0',z_0'')$$

and expand $f'(z',z'')$ and $f''(z',z'')$ in Taylor series:

$$f'(z',z'') - f'(z_0',z_0'') = \frac{\partial f'}{\partial z'}\Big|_{z_0}(z'-z_0') + \frac{\partial f'}{\partial z''}\Big|_{z_0}(z''-z_0'') + \dots$$

$$f''(z',z'') - f''(z_0',z_0'') = \frac{\partial f''}{\partial z'}\Big|_{z_0}(z'-z_0') + \frac{\partial f''}{\partial z''}\Big|_{z_0}(z''-z_0'') + \dots$$

$$\rightarrow \underline{\frac{f(z)-f(z_0)}{z-z_0}} = \frac{1}{z-z_0}\Big[\frac{\partial f'}{\partial z'}\Big|_{z_0}(z'-z_0') + \frac{\partial f'}{\partial z''}\Big|_{z_0}(z''-z_0'') + i\frac{\partial f''}{\partial z'}\Big|_{z_0}(z'-z_0') + i\frac{\partial f''}{\partial z''}\Big|_{z_0}(z''-z_0'')\Big]$$

1

+ (terms that vanish as $z \to z_0$)

$$\overset{\text{CR-eqs}}{=} \frac{1}{z-z_0}\Big[\frac{\partial f'}{\partial z'}\Big|_{z_0}(z'-z_0'+iz''-iz_0'') + \frac{\partial f'}{\partial z''}(z''-z_0''-iz'+iz_0')\Big]$$

$$= \frac{\partial f'}{\partial z'}\Big|_{z_0}\frac{z-z_0}{z-z_0} - i\frac{\partial f'}{\partial z''}\Big|_{z_0}\frac{z-z_0}{z-z_0}$$

1

$$= \Big(\frac{\partial f'}{\partial z'} - i\frac{\partial f'}{\partial z''}\Big)\Big|_{z_0} \overset{\text{CR-eqs}}{=} \underline{\frac{\partial f'}{\partial z'} + i\frac{\partial f''}{\partial z'}} + \text{(terms that vanish as } z\to z_0)$$

1

But our premise was that the CR-eqs hold

→ the rhs exists

→ the limit $\lim\limits_{z\to 0}\frac{f(z)-f(z_0)}{z-z_0} = \frac{df}{dz}\Big|_{z_0}$ exists and is independent of how z_0 is approached.

1 → <u>The Cauchy-Riemann eqs are sufficient</u>