II.2.1. Lindhard function

Consider the function $F : \mathbb{C} \to \mathbb{C}$ (which plays an important role in the theory of many-electron systems) defined by

$$F(z) = \frac{i}{\pi} \log\left(\frac{z-1}{z+1}\right)$$

The *spectral density* $F'' : \mathbb{R} \to \mathbb{C}$ and the *reactive part* $F' : \mathbb{R} \to \mathbb{C}$ of F are defined by

$$F''(\omega) := \frac{1}{2i}\left[F(\omega + i0) - F(\omega - i0)\right] \qquad , \qquad F'(\omega) := \frac{1}{2}\left[F(\omega + i0) + F(\omega - i0)\right]$$

where $F(\omega \pm i0) := \lim_{\epsilon\to 0} F(\omega \pm i\epsilon)$.

a) Determine F'' and F' explicitly, and plot $-iF''$ and $-iF'$ for $-3 < \omega < 3$.

b) Show that

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi}\, \frac{F''(\omega)}{\omega - z} = F(z)$$

(5 points)

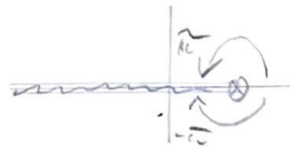
Solution

II.2.1.) $F(z) = \frac{i}{\pi} \log \frac{z-1}{z+1} = \frac{i}{\pi}\left[\log(z-I) - \log(z+I)\right]$

$F''(\omega) = \frac{1}{2i}\left[F(\omega+i0) - F(\omega-i0)\right]$ spectral density

$F'(\omega) = \frac{1}{2}\left[F(\omega+i0) + F(\omega-i0)\right]$ reactive part

a) $\log(\omega+i0-I) = \begin{cases} \log(\omega-I) & \text{if } \omega > I \\ \log|\omega-I| + i\pi & \text{if } \omega < I \end{cases}$



$\log(\omega-i0-I) = \begin{cases} \log(\omega-I) & \text{if } \omega > I \\ \log|\omega-I| - i\pi & \text{if } \omega < I \end{cases}$

$\log(\omega+i0+I) = \begin{cases} \log(\omega+I) & \text{if } \omega > -I \\ \log|\omega+I| + i\pi & \text{if } \omega < -I \end{cases}$

$\log(\omega-i0+I) = \begin{cases} \log(\omega+I) & \text{if } \omega > -I \\ \log|\omega+I| - i\pi & \text{if } \omega < -I \end{cases}$

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$$\Rightarrow \underline{F''(\omega)} = \frac{i}{\pi}\cdot\frac{1}{2i}\Big\{\Theta(\omega>I)\left[\cancel{\log(\omega-I)} - \cancel{\log(\omega+I)} - \cancel{\log(\omega-I)} + \cancel{\log(\omega+I)}\right]$$

$$+\,\Theta(-I<\omega<I)\left[\cancel{\log|\omega-I|} + i\pi - \cancel{\log(\omega+I)} - \cancel{\log|\omega-I|} + i\pi + \cancel{\log(\omega+I)}\right]$$

$$+\,\Theta(\omega<-I)\Big[\cancel{\log|\omega-I|} + \cancel{i\pi} - \cancel{\log|\omega+I|} - \cancel{i\pi} - \cancel{\log|\omega-I|} + \cancel{i\pi} + \cancel{\log|\omega+I|} + \cancel{i\pi}\Big]\Big\}$$

$$= \frac{1}{2\pi}\,\Theta(\omega^2<I)\,2i\pi$$

$$= \underline{\underline{i\,\Theta(\omega^2<I)}}$$

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$$\underline{F'(\omega)} = \frac{i}{\pi}\frac{1}{2}\Big\{ \Theta(\omega>1)\big[\log(\omega-1) - \log(\omega+1) + \log(\omega-1) - \log(\omega+1)\big]$$

$$+ \Theta(-1<\omega<1)\big[\log|\omega-1| + \cancel{i\pi} - \log(\omega+1) + \log|\omega-1| - \cancel{i\pi} - \log(\omega+1)\big]$$

$$+ \Theta(\omega<-1)\big[\log|\omega-1| + \cancel{i\pi} - \log|\omega+1| - \cancel{i\pi} + \log|\omega-1| - \cancel{i\pi} - \log|\omega+1| + \cancel{i\pi}\big]\Big\}$$

$$= \frac{i}{2\pi}\Big\{\Theta(\omega>1)\, 2\log\frac{\omega-1}{\omega+1} + \Theta(-1<\omega<1)\, 2\log\frac{|\omega-1|}{\omega+1} + \Theta(\omega<-1)\, 2\log\left|\frac{\omega-1}{\omega+1}\right|\Big\}$$

$$= \underline{\frac{i}{\pi}\log\left|\frac{\omega-1}{\omega+1}\right|}$$

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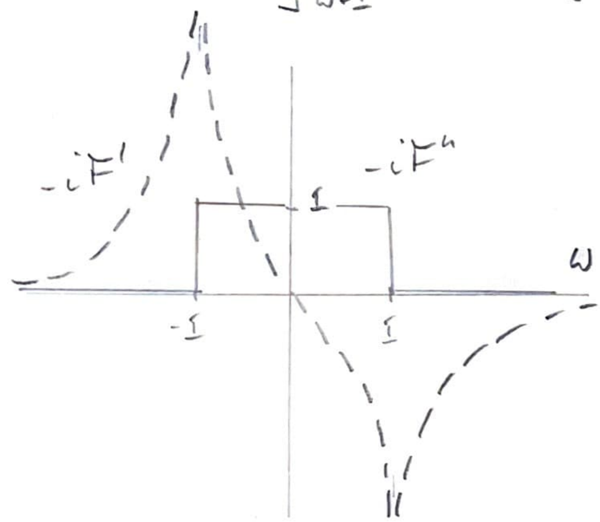

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b) $$\underline{\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{F''(\omega)}{\omega-z}} = \frac{i}{\pi}\int_{-1}^{1} d\omega\,\frac{1}{\omega-z} = \frac{i}{\pi}\log(\omega-z)\Big|_{-1}^{1}$$

$$= \frac{i}{\pi}\log\frac{1-z}{-1-z} = \underline{\frac{i}{\pi}\log\frac{z-1}{z+1}} = \underline{F(z)}$$

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