I.6.3. Tensor products, and tensor traces

Prove Propositions 5 and 6 from ch. I §6.3.

(4 points)

Solution

1.6.1) Consider the tensor product

$$u^{i_1 \dots i_{N+n}} = s^{i_1 \dots i_N}\, t^{i_{N+1} \dots i_{N+n}}$$

Let both s and t be tensors. Then

$$\tilde{u}^{i_1 \dots i_{N+n}} = \tilde{s}^{i_1 \dots i_N}\, \tilde{t}^{i_{N+1} \dots i_{N+n}}$$

$$= D^{i_1}{}_{j_1} \dots D^{i_N}{}_{j_N}\, s^{j_1 \dots j_N}\, D^{i_{N+1}}{}_{j_{N+1}} \dots D^{i_{N+n}}{}_{j_{N+n}}\, t^{j_{N+1} \dots j_{N+n}}$$

$$= D^{i_1}{}_{j_1} \dots D^{i_{N+n}}{}_{j_{N+n}}\, u^{j_1 \dots j_{N+n}} \qquad (*)$$

$\rightarrow$ u is a rank-$(N+n)$ tensor

Now suppose s is a pseudotensor and t a tensor, or vice versa. Then the rhs of $(*)$ gets multiplied by $\det D$ $\rightarrow$ u is a pseudotensor

(1)

(1) If both s and t are pseudotensors, then the rhs of $(*)$ gets multiplied by $(\det D)^2 = 1$ $\rightarrow$ u is a tensor

This proves Prop.5 from $ 6.3

Now consider

$$u^{\lambda_1 \dots \lambda_N} = t_i{}^{i \lambda_1 \dots \lambda_N} = g_{ij}\, t^{ij \lambda_1 \dots \lambda_N}$$

Let t be a tensor. Then

$$\tilde{u}^{\lambda_1 \dots \lambda_N} = \tilde{g}_{ij}\, \tilde{t}^{ij \lambda_1 \dots \lambda_N}$$

Problem 25 $\Rightarrow$

(1) $$= ((D^{-1})^T)_i{}^o\, ((D^{-1})^T)_j{}^p\, g_{op}\, D^i{}_m D^j{}_n D^{\lambda_1}{}_{\ell_1} \dots D^{\lambda_N}{}_{\ell_N}\, t^{mn \ell_1 \dots \ell_N}$$

$$= \underbrace{(D^{-1})^o{}_i D^i{}_m}_{=\delta^o{}_m}\, \underbrace{(D^{-1})^p{}_j D^j{}_n}_{=\delta^p{}_n}\, D^{\lambda_1}{}_{\ell_1} \dots D^{\lambda_N}{}_{\ell_N}\, g_{op}\, t^{mn \ell_1 \dots \ell_N}$$

$$= D^{k_1}{}_{l_1} \cdots D^{k_N}{}_{l_N} \, g_{mn} \, t^{mn l_1 \dots l_N}$$

$$= D^{k_1}{}_{l_1} \cdots D^{k_N}{}_{l_N} \, u^{l_1 \dots l_N} \qquad (**)$$

$\leadsto$ u is a tensor of rank N

If t is a pseudotensor, then the rhs of (**) gets multiplied by det D

$\leadsto$ u is a pseudotensor of rank N

1 This proves Prop. 6 from $ 6.3